A New Type of $\xi$-Open Sets Based on Operations

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ABSTRACT

The aim of this paper is to introduce a new type of $\xi$-open sets in topological spaces which is called $\xi\gamma$-open sets and we study some of their basic properties and characteristics.

Keywords: Open sets, $\xi$-Space.

1. Introduction

Ogata [9], introduced the concept of an operation on a topology, then after authors defined some other types of sets such as $\gamma$-open [9], $\gamma$-semi-open [6], $\gamma$-pre semi-open [6] and $\gamma$-$\beta$-open [1] sets in a topological space by using operations. In [4] the concept of $\xi$-open set in a topological space is introduced and studied.

The purpose of this paper, is to introduce a new class of $\xi$-open sets namely $\xi\gamma$-open sets and establish basic properties and relationships with other types of sets, also we define the notions of $\xi\gamma$-neighbourhood, $\xi\gamma$-derived, $\xi\gamma$-closure and $\xi\gamma$-interior of a set and give some of their properties which are mostly analogous to those properties of open sets. Throughout this paper, $(X, \tau)$ or(briefly, X) mean a topological space on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space X, Cl(A) and Int(A) are denoted respectively the closure and interior of A.

2. Preliminaries.

We start this section by introducing some definitions and results concerning sets and spaces which will be used later.

Definition 2.1. A subset A of a space $(X, \tau)$ is called:
1) semi-open [7], if $A \subseteq \text{Cl} (\text{Int} (A))$.
2) regular open [2], if $A = \text{Int} (\text{Cl} (A))$.

The complement of semi-open (resp., regular open, preopen and $\alpha$-open) set is said to be semi-closed (resp., regular closed, preclosed and $\alpha$-closed).

**Definition 2.2.** [4] An open subset $U$ of a space $X$ is called $\xi$-open if for each $x \in U$, there exists a semi-closed set $F$ such that $x \in F \subseteq U$. The family of all $\xi$-open subsets of a topological space $(X, \tau)$ is denoted by $\xi \text{O}(X, \tau)$ or (briefly $\xi \text{O}(X)$). The complement of each $\xi$-open set is called $\xi$-closed set. The family of all $\xi$-closed subsets of a topological space $(X, \tau)$ is denoted by $\xi \text{C}(X, \tau)$ or (briefly $\xi \text{C}(X)$).

**Definition 2.3.** [5] Let $(X, \tau)$ be a topological space. An operation $\gamma$ on the topology $\tau$ is a mapping from $\tau$ into power set $P(X)$ such that $V \subseteq \gamma (V)$ for each $V \in \tau$, where $\gamma (V)$ denotes the value of $\gamma$ at $V$.

**Definition 2.4.** [8]
1) A subset $A$ of a topological space $(X, \tau)$ is called $\gamma$-open set if for each $x \in A$ there exists an open set $U$ such that $x \in U$ and $\gamma (U) \subseteq A$. Clearly $\gamma \tau \subseteq \tau$.
2) The point $x \in X$ is in the $\gamma$-closure of a set $A \subseteq X$, if $\gamma (U) \cap A \neq \emptyset$, for each open set $U$ containing $x$. The $\gamma$-closure of a set $A$ is denoted by $\gamma \text{Cl}(A)$.
3) Let $(X, \tau)$ be a topological space and $A$ be subset of $X$, then $\tau_\gamma \text{Cl}(A) = \cap \{ F: A \subseteq F, X \setminus F \in \tau_\gamma \}$.

**Definition 2.5.** [11] Let $(X, \tau)$ be a topological space and $A$ be subset of $X$, then $\tau_\gamma \text{Int}(A) = \cup \{ U: U$ is $\gamma$-open set and $U \subseteq A \}$.

**Definition 2.6.** [1] Let $(X, \tau)$ be a topological space with an operation $\gamma$ on $\tau$:
1) The $\gamma$-derived set of $A$ is defined by $\{ x: \text{for every } \gamma$-open set $U$ containing $x$, $U \cap (A \setminus \{x\}) \neq \emptyset \}$
2) The $\gamma$-boundary of $A$ is defined as $\tau_\gamma \text{Cl}(A) \cap \tau_\gamma \text{Cl}(X \setminus A)$.

**Definition 2.7.** [4] Let $(X, \tau)$ be a topological space and $A \subseteq X$, then:
1) $\xi$-interior of $A$ is the union of all $\xi$-open sets contained in $A$.
2) $\xi$-closure of $A$ is the intersection of all $\xi$-closed sets containing $A$.

**Lemma 2.8.** [4]
1) Let $(Y, \tau_Y)$ be a subspace of $(X, \tau)$. If $F \in \text{SC}(X, \tau)$ and $F \subseteq Y$, then $F \in \text{SC}(Y, \tau_Y)$.
2) Let $(Y, \tau_Y)$ be a subspace of $(X, \tau)$. If $F \in \text{SC}(Y, \tau_Y)$ and $Y \subseteq \text{SC}(X, \tau)$, then $F \in \text{SC}(X, \tau)$.

**Lemma 2.9** [4]
1) Let $Y$ be a regular open subspace of a space $X$. If $G \in \xi \text{O}(Y)$, then $G \in \xi \text{O}(X)$.
2) Let $Y$ be a subspace of a space $X$ and $Y \subseteq \text{SC}(X)$. If $G \in \xi \text{O}(X)$ and $G \subseteq Y$, then $G \in \xi \text{O}(Y)$.

3. $\xi_\gamma$-Open Sets

In this section, a new class of $\xi$-open sets called $\xi_\gamma$-open sets in topological spaces is introduced. We define $\gamma$ to be a mapping on $\xi \text{O}(X)$ into $P(X)$ and we say that $\gamma: \xi \text{O}(X) \rightarrow P(X)$ is an $\xi$-operation on $\xi \text{O}(X)$ if $V \subseteq \gamma (V)$, for each $V \in \xi \text{O}(X)$.
**Definition 3.1** A subset $A$ of a space $X$ is called $\gamma$-open if for each point $x \in A$, there exist an $\xi$-open set $U$ such that $x \in U \subseteq \gamma(U) \subseteq A$.

The family of all $\gamma$-open subset of a topological space $(X, \tau)$ is denoted by $\gamma O(X, \tau)$ or (briefly $\gamma O(X)$).

A subset $B$ of a space $X$ is called $\gamma$-closed if $X \setminus B$ is $\gamma$-open. The family of all $\gamma$-closed subsets of a topological space $(X, \tau)$ is denoted by $\gamma C(X, \tau)$ or (briefly $\gamma C(X)$).

**Remark 3.2** From the definition of the operation $\gamma$, it is clear that $\gamma(X) = X$ for any $\xi$-operation $\gamma$. For competence, it is assumed that $\gamma(\emptyset) = \emptyset$ for any $\xi$-operation $\gamma$.

**Remark 3.3** It is clear from the definition that every $\xi$-open subset of a space $X$ is $\gamma$-open, but the converse is not true in general as shown in the following example:

**Example 3.5.** Consider $X = \{a, b, c, d\}$ with the topology $\tau = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$. Define an $\xi$-operation $\gamma$ by

$$
\gamma(A) =
\begin{cases}
A & \text{if } a \in A \\
X & \text{if } a \notin A
\end{cases}
$$

Then $\{c\}$ is open and $\xi$-open but $\{c\} \notin \xi O(X)$.

**Proposition 3.6.** Every $\xi$-open set of a space $X$ is $\gamma$-open.

**Proof.** Let $A$ be $\xi$-open in a topological space $(X, \tau)$, then for each point $x \in A$, there exists an $\xi$-open set $U$ such that $x \in U \subseteq \gamma(U) \subseteq A$. Since every $\xi$-open set is open, this implies that $A$ is a $\gamma$-open set.

The following example shows that the converse of the above proposition is not true in general.

**Example 3.7** Consider $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{a\}\}$. Define an $\xi$-operation $\gamma$ by $\gamma(A) = A$ for any subset $A$ of $X$. Then, $\{a\}$ is $\gamma$-open set but not $\xi$-open set. Hence, it is not $\xi$-open.

The following result shows that any union of $\xi$-open sets in a topological space $(X, \tau)$ is $\xi$-open.

**Proposition 3.8** Let $\{A_{\lambda}\}_{\lambda \in \Lambda}$ be a collection of $\xi$-open sets in a topological space $(X, \tau)$. Then, $\bigcup_{\lambda \in \Lambda} A_{\lambda}$ is $\xi$-open.

**Proof.** Let $x \in \bigcup_{\lambda \in \Lambda} A_{\lambda}$, then $x \in A_{\lambda}$ for some $\lambda \in \Lambda$. Since, $A_{\lambda}$ is an $\xi$-open set, then there exists an $\xi$-open set $U$ containing $x$ and $\gamma(U) \subseteq A_{\lambda} \subseteq \bigcup_{\lambda \in \Lambda} A_{\lambda}$. Therefore, $\bigcup_{\lambda \in \Lambda} A_{\lambda}$ is an $\xi$-open set in a topological space $(X, \tau)$.

The following example shows that the intersection of two $\xi$-open sets need not be an $\xi$-open set.

**Example 3.9** Consider $X = \{a, b, c\}$ with discrete topology on $X$. Define an $\xi$-operation $\gamma$ by
\[
\gamma(A) = \begin{cases} 
\{a, b\} & \text{if } A = \{a\} \text{ or } \{b\} \\
A & \text{otherwise}
\end{cases}
\]

Let \(A = \{a, b\}\) and \(B = \{b, c\}\), it is clear that \(A\) and \(B\) are \(\xi\)-open sets, but \(A \cap B = \{b\}\) is not \(\xi\)-open set.

From the above example, we notice that the family of all \(\xi\)-open subsets of a space \(X\) is a supratopology and need not be a topology in general.

**Proposition 3.10** The set \(A\) is \(\xi\)-open in the space \((X, \tau)\) if and only if for each \(x \in A\), there exists an \(\xi\)-open set \(B\) such that \(x \in B \subseteq A\).

**Proof.** Suppose that \(A\) is an \(\xi\)-open set in the space \((X, \tau)\). Then, for each \(x \in A\), put \(B = A\) is an \(\xi\)-open set such that \(x \in B \subseteq A\).

**Conversely**, suppose that for each \(x \in A\), there exists an \(\xi\)-open set \(B_x\) such that \(x \in B_x \subseteq A\), thus \(A = \bigcup B_x\) where \(B_x \in \xi O(X)\) for each \(x \in A\). Therefore, \(A\) is \(\xi\)-open set.

**Definition 3.11** Let \((X, \tau)\) be a topological space. A mapping \(\gamma : \xi O(X) \rightarrow P(X)\) is said to be:

1) \(\xi\)-identity on \(\xi O(X)\) if \(\gamma(A) = A\) for all \(A \in \xi O(X)\).
2) \(\xi\)-monotone on \(\xi O(X)\) if for all \(A, B \in \xi O(X)\), \(A \subseteq B\) implies \(\gamma(A) \subseteq \gamma(B)\).
3) \(\xi\)-idempotent on \(\xi O(X)\) if \(\gamma(\gamma(A)) = \gamma(A)\) for all \(A \in \xi O(X)\).
4) \(\xi\)-additive on \(\xi O(X)\) if \(\gamma(A \cup B) = \gamma(A) \cup \gamma(B)\) for all \(A, B \in \xi O(X)\).

If \(\bigcup i \in I \gamma(A_i) \subseteq \gamma(\bigcup i \in I A_i)\) for any collection \(\{A_i\}_{i \in I} \subseteq \xi O(X)\), then \(\gamma\) is said to be \(\xi\)-subadditive on \(\xi O(X)\).

**Proposition 3.12.** Let \(\gamma\) be an \(\xi\)-operation. Then, \(\gamma\) is \(\xi\)-monotone on \(\xi O(X)\) if and only if \(\gamma\) is subadditive on \(\xi O(X)\).

**Proof.** Let \(\gamma\) be \(\xi\)-monotone on \(\xi O(X)\) and let \(\{A_i\}_{i \in I} \subseteq \xi O(X)\). Then, for each \(i \in I\), \(\gamma(A_i) \subseteq \gamma(\bigcup i \in I A_i)\) and thus \(\bigcup i \in I \gamma(A_i) \subseteq \gamma(\bigcup i \in I A_i)\). Therefore, \(\gamma\) is \(\xi\)-subadditive on \(\xi O(X)\).

**Conversely**, if \(\gamma\) is subadditive on \(\xi O(X)\), and \(A, B \in \xi O(X)\) with \(A \subseteq B\), then \(\gamma(A) \subseteq \gamma(A) \cup \gamma(B) \subseteq \gamma(A \cup B) = \gamma(B)\). Thus, \(\gamma\) is \(\xi\)-monotone on \(\xi O(X)\).

The following result shows that if \(\gamma\) is \(\xi\)-monotone, then the family of \(\xi\)-open sets is a topology on \(X\).

**Proposition 3.13** If \(\gamma\) is \(\xi\)-monotone, then the family of \(\xi\)-open sets is a topology on \(X\).

**Proof.** Clearly \(\phi, X \in \xi O(X)\) and by Proposition 3.8, the union of any family \(\xi\)-open sets is \(\xi\)-open set. To complete the proof, it is enough to show that the finite intersection of \(\xi\)-open sets is an \(\xi\)-open set. Let \(A\) and \(B\) be two \(\xi\)-open sets and let \(x \in A \cap B\), then \(x \in A\) and \(x \in B\), so there exists \(\xi\)-open sets namely \(U\) and \(V\) such that \(x \in U \subseteq \gamma(U) \subseteq A\) and \(x \in V \subseteq \gamma(V) \subseteq B\), since \(U\) and \(V\) are \(\xi\)-open sets then \(U \cap V\) is \(\xi\)-open, but \(U \cap V \subseteq U\) and \(U \cap V \subseteq V\), but \(\gamma\) is \(\xi\)-monotone operation, therefore \(\gamma(U \cap V) \subseteq \gamma(U) \cap \gamma(V) \subseteq A \cap B\). Thus, \(A \cap B\) is an \(\xi\)-open set. This completes the proof.

**Proposition 3.14** Let \(Y\) be a semi-closed subspace of a space \(X\). If \(A \in \xi O(X, \tau)\) and \(A \subseteq Y\), then \(A \in \xi O(Y, \tau_Y)\), where \(\gamma\) is \(\xi\)-identity on \(\xi O(Y)\).
Proof. Let \( A \in \xi O(X, \tau) \), then \( A \in \xi O(X, \tau) \) and for each \( x \in A \) there exists an \( \xi \)-open set \( U \) in \( X \) such that \( x \in U \subseteq \gamma(U) \subseteq A \). Since, \( A \in \xi O(X, \tau) \) and \( A \subseteq Y \), where \( Y \) is semi-closed in \( X \), then by Proposition 2.14, \( U \in \xi Y/O(Y, \tau) \). Hence, \( A \in \xi Y/O(Y, \tau) \).

**Proposition 3.15** Let \( Y \) be a regular open subspace of a space \((X, \tau)\) and \( \gamma \) is an \( \xi \)-identity on \( \xi O(X) \). If \( A \in \xi Y/O(Y, \tau) \) and \( Y \in \xi O(X, \tau) \), then \( A \in \xi Y/O(X, \tau) \).

**Proof.** Let \( A \in \xi Y/O(Y, \tau) \), then \( A \in \xi Y/O(Y, \tau) \) and for each \( x \in A \) there exists an \( \xi \)-open set \( U \) in \( Y \) such that \( x \in U \subseteq \gamma(Y) \subseteq A \). Since, \( Y \in \xi O(X, \tau) \) and \( A \in \xi O(Y, \tau) \), then by Proposition 2.13, \( U \in \xi O(X, \tau) \). Hence, \( A \in \xi O(X, \tau) \).

4. Other Properties of \( \xi \)-Open Sets

In this section, we define and study some properties of \( \xi \)-neighbourhood of a point, \( \xi \)-derived, \( \xi \)-closure and \( \xi \)-interior of sets via \( \xi \)-open sets.

**Definition 4.1** Let \((X, \tau)\) be a topological space and \( x \in X \), then a subset \( N \) of \( X \) is said to be \( \xi \)-neighbourhood of \( x \), if there exists an \( \xi \)-open set \( U \) in \( X \) such that \( x \in U \subseteq N \).

**Proposition 4.2** Let \((X, \tau)\) be a topological space. A subset \( A \) of \( X \) is \( \xi \)-open if and only if it is an \( \xi \)-neighbourhood of each of its points.

**Proof.** Let \( A \subseteq X \) be an \( \xi \)-open set. Since, for every \( x \in A \), \( x \in A \subseteq A \) and \( A \) is \( \xi \)-open, then \( A \) is an \( \xi \)-neighbourhood of each of its points.

**Conversely,** suppose that \( A \) is an \( \xi \)-neighbourhood of each of its points. Then, for each \( x \in A \), there exists \( B_x \in \xi Y/O(X) \) such that \( B_x \subseteq A \). Then, \( A = \cup \{ B_x : x \in A \} \). Since, each \( B_x \) is \( \xi \)-open, it follows that \( A \) is an \( \xi \)-open set.

**Definition 4.3** Let \((X, \tau)\) be a topological space with an operation \( \gamma \) on \( \xi O(X) \). A point \( x \in X \) is said to be \( \xi \)-limit point of a set \( A \) if for each \( \xi \)-open set \( U \) containing \( x \), then \( U \cap (A \setminus \{x\}) \neq \emptyset \). The set of all \( \xi \)-limit points of \( A \) is called \( \xi \)-derived set of \( A \) and denoted by \( \xi Y/D(A) \).

**Proposition 4.5** Let \( A \) and \( B \) be subsets of a space \( X \). If \( A \subseteq B \), then \( \xi Y/D(A) \subseteq \xi Y/D(B) \).

**Proof.** Obvious.

Some properties of \( \xi \)-derived sets are stated in the following proposition.

**Proposition 4.6** Let \( A \) and \( B \) be any two subsets of a space \( X \), and \( \gamma \) be an operation on \( \xi O(X) \). Then, we have the following properties:

1) \( \xi Y/D(\emptyset) = \emptyset \).
2) If \( x \in \xi Y/D(A) \), then \( x \in \xi Y/D(A \setminus \{x\}) \).
3) \( \xi Y/D(A) \cup \xi Y/D(B) \subseteq \xi Y/D(A \cup B) \).
4) \( \xi Y/D(A \cap B) \subseteq \xi Y/D(A) \cap \xi Y/D(B) \).
5) \( \xi Y/D(\xi Y/D(A)) \setminus A \subseteq \xi Y/D(A) \).
6) \( \xi Y/D(A) \cup \xi Y/D(A) \subseteq A \cup \xi Y/D(A) \).

**Proof.** Straightforward.

In general, the equalities of (3), (4) and (6) of the above proposition do not hold, as is shown in the following examples.
Example 4.7 Consider $X = \{a, b, c\}$ with discrete topology on $X$. Define an operation $\gamma$ on $\xi O(X)$ by

$$
\gamma(A) = \begin{cases}
A & \text{if } A = \{b\} \text{ or } \{a, b\} \text{ or } \{a, c\} \\
X & \text{otherwise}
\end{cases}
$$

Now, if $A = \{a, b\}$ and $B = \{a, c\}$, then $\xi_\gamma(D(A)) = \{c\}$, $\xi_\gamma(D(B)) = \{c\}$ and $\xi_\gamma(D(A \cup B)) = \{a, c\}$, where $A \cup B = X$, this implies that $\xi_\gamma(D(A) \cup \xi_\gamma(D(B)) \neq \xi_\gamma(D(A \cup B))$.

Example 4.8 Consider $X = \{a, b, c, d\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define an operation $\gamma$ on $\xi O(X)$ by

$$
\gamma(A) = \begin{cases}
A & \text{if } b \in A \\
X & \text{if } b \notin A
\end{cases}
$$

Now, if we let $A = \{a, b\}$ and $B = \{c, d\}$, then $\xi_\gamma(D(A)) = \{a, c, d\}$, $\xi_\gamma(D(B)) = \{d\}$, hence $\xi_\gamma(D(A) \cap \xi_\gamma(D(B)) = \{d\}$, but $\xi_\gamma(D(A \cap B)) = \emptyset$, where $A \cap B = \emptyset$, this implies that $\xi_\gamma(D(A) \cap \xi_\gamma(D(B)) \neq \xi_\gamma(D(A \cap B))$. Also $\xi_\gamma(D(A)) = \{d\}$, therefore $\xi_\gamma(D(A)) \not\subseteq \xi D(A)$.

Definition 4.9 Let $A$ be a subset of a topological space $(X, \tau)$ and $\gamma$ be an operation on $\xi O(X)$. The intersection of all $\xi_\gamma$-closed sets containing $A$ is called the $\xi_\gamma$-closure of $A$ and denoted by $\xi_\gamma Cl(A)$.

Here, we introduce some properties of $\xi_\gamma$-closure of the sets.

Proposition 4.10 Let $(X, \tau)$ be a topological space and $\gamma$ be an operation on $\xi O(X)$. For any subsets $A$ and $B$ of $X$, we have the following:

1) $A \subseteq \xi_\gamma Cl(A)$.
2) $\xi_\gamma Cl(A)$ is an $\xi_\gamma$-closed set in $X$.
3) $A$ is an $\xi_\gamma$-closed set if and only if $A = \xi_\gamma Cl(A)$.
4) $\xi_\gamma Cl(\emptyset) = \emptyset$ and $\xi_\gamma Cl(X) = X$.
5) $\xi_\gamma Cl(A) \cup \xi_\gamma Cl(B) \subseteq \xi_\gamma Cl(A \cup B)$.
6) $\xi_\gamma Cl(A \cap B) \subseteq \xi_\gamma Cl(A) \cap \xi_\gamma Cl(B)$.

Proof. They are obvious.

In general, the equalities of (5) and (6) of the above proposition does not hold, as is shown in the following examples:

Example 4.11 Consider $X = \{a, b, c\}$ with discrete topology on $X$. Define an operation $\gamma$ on $\xi O(X)$ by

$$
\gamma(A) = \begin{cases}
A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \\
X & \text{otherwise}
\end{cases}
$$

Then, $\xi_\gamma O(X) = \{\emptyset, X, \{a, b\}, \{a, c\}\}$. Now, if we let $A = \{b\}$ and $B = \{c\}$, then $\xi_\gamma Cl(A) = A$, $\xi_\gamma D(B) = B$ and $\xi_\gamma Cl(A \cup B) = X$, where $A \cup B = \{b, c\}$, this implies that $\xi_\gamma Cl(A) \cup \xi_\gamma Cl(B) \neq \xi_\gamma Cl(A \cup B)$.

Example 4.12 Consider $X = \{a, b, c, d\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define an operation $\gamma$ on $\xi O(X)$ by.
\[ \gamma(A) = \begin{cases} A & \text{if } b \in A \\ X & \text{if } b \notin A \end{cases} \]

It is clear that \( \xi_r\text{-}O(X) = \{ \emptyset, X, \{b\}, \{a, b\}, \{a, b, c\}\} \). Now, if we let \( A = \{c\} \) and \( B = \{d\} \), then \( \xi_r\text{-}O(A) = \{c, d\} \) and \( \xi_r\text{-}O(B) = \{d\} \), hence \( \xi_r\text{-}O(A \cap B) = \{d\} \), but \( \xi_r\text{-}O(A \cap B) = \emptyset \), this implies that \( \xi_r\text{-}O(A \cap B) \neq \xi_r\text{-}O(A) \cap \xi_r\text{-}O(B) \).

Now, if we let \( A = \{b\} \), we see that \( \xi_r\text{-}O(A) = \{b, d\} \), but \( \xi_r\text{-}O(A) = \emptyset \). Hence, \( \xi_r\text{-}O(A) \neq \xi_r\text{-}O(A) \).

**Proposition 4.18** A subset \( A \) of a topological space \( X \) is an \( \xi_r\text{-}O(X) \). The union of all \( \xi_r\text{-}O(X) \) is \( \emptyset \). The proof of the following two results is obvious.

**Proposition 4.14** Let \( A \) be a subset of a topological space \( (X, \tau) \) and \( \gamma \) be an \( \xi_r\text{-}O(X) \). Then, \( x \in \xi_r\text{-}O(A) \) if and only if for every \( \xi_r\text{-}O(A) \) containing \( x \), \( A \) is an \( \xi_r\text{-}O(A) \) set.

**Proof.** Let \( x \in \xi_r\text{-}O(A) \) and suppose that \( A \cap V = \emptyset \), for some \( \xi_r\text{-}O(A) \) containing \( x \). Then, \( (XV) = \xi_r\text{-}O(A) \) and \( A \subseteq (XV) \), thus \( \xi_r\text{-}O(A) \subseteq (XV) \). But, this implies that \( x \in (XV) \) which is contradiction. Therefore, \( A \cap V \neq \emptyset \).

Conversely, Let \( A \subseteq X \) and \( x \in X \) such that for each \( \xi_r\text{-}O(A) \) containing \( x \), \( A \cap V \neq \emptyset \). If \( x \notin \xi_r\text{-}O(A) \), there exists an \( \xi_r\text{-}O(A) \) such that \( A \subseteq F \). Then, \( (X \cap F) \) is an \( \xi_r\text{-}O(A) \) set with \( x \in (X \cap F) \), and thus \( (\cap F \cap A \neq \emptyset \), which is contradiction.

The proof of the following two results is obvious.

**Proposition 4.15** Let \( A \) be a subset of a topological space \( (X, \tau) \) and \( \gamma \) be an \( \xi_r\text{-}O(X) \). Then, \( \xi_r\text{-}O(A) = A \cup \xi_r\text{-}O(D(A)) \).

**Proposition 4.16** If \( A \) and \( B \) are subsets of a space \( X \) with \( A \subseteq B \). Then, \( \xi_r\text{-}O(A) \subseteq \xi_r\text{-}O(B) \).

**Definition 4.4** Let \( A \) be a subset of a topological space \( (X, \tau) \) and \( \gamma \) be an operation on \( \xi_r\text{-}O(X) \). The union of all \( \xi_r\text{-}O(A) \) contained in \( A \) is called the \( \xi_r\text{-}O(A) \) and denoted by \( \xi_r\text{-}O(A) \).

Here, we introduce some properties of \( \xi_r\text{-}O(A) \) of the sets.

**Proposition 4.17** Let \( (X, \tau) \) be a topological space and \( \gamma \) be an operation on \( \xi_r\text{-}O(X) \). For any subsets \( A \) and \( B \) of \( X \), we have the following:

1) \( \xi_r\text{-}O(A) \) is an \( \xi_r\text{-}O(B) \) set in \( X \).
2) \( A \) is an \( \xi_r\text{-}O(B) \) if and only if \( A = \xi_r\text{-}O(B) \).
3) \( \xi_r\text{-}O(A) \cap \xi_r\text{-}O(B) \) = \( \xi_r\text{-}O(A) \cap \xi_r\text{-}O(B) \).
4) \( \xi_{\gamma}\text{-Int}(\phi) = \phi \) and \( \xi_{\gamma}\text{-Int}(X) = X \).
5) \( \xi_{\gamma}\text{-Int}(A) \subseteq A \).
6) If \( A \subseteq B \), then \( \xi_{\gamma}\text{-Int}(A) \subseteq \xi_{\gamma}\text{-Int}(B) \).
7) \( \xi_{\gamma}\text{-Int}(A) \cup \xi_{\gamma}\text{-Int}(B) \subseteq \xi_{\gamma}\text{-Int}(A \cup B) \).
8) \( \xi_{\gamma}\text{-Int}(A \cap B) \subseteq \xi_{\gamma}\text{-Int}(A) \cap \xi_{\gamma}\text{-Int}(B) \).

**Proof.** Straightforward.

In general, the equalities of (7) and (8) of the above proposition do not hold, as is shown in the following examples:

**Example 4.19** Consider \( X = \{a, b, c, d\} \) with the topology \( \tau = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \). Define an \( \xi\text{-operation} \gamma \) by

\[
\gamma(A) = \begin{cases} 
A & \text{if } b \in A \\
X & \text{if } b \notin A
\end{cases}
\]

It is clear that \( \xi_{\gamma}\text{-O}(X) = \{\phi, X, \{b\}, \{a, b\}, \{a, b, c\}\} \). Now, if we let \( A = \{a\} \) and \( B = \{b\} \), then \( \xi_{\gamma}\text{-Int}(A) = \phi \) and \( \xi_{\gamma}\text{-Int}(B) = \{b\} \), hence \( \xi_{\gamma}\text{-Int}(A) \cup \xi_{\gamma}\text{-Int}(B) = \{b\} \), but \( \xi_{\gamma}\text{-Int}(A \cup B) = \{a, b\} \), where \( A \cup B = \{a, b\} \), this implies that \( \xi_{\gamma}\text{-Int}(A \cup B) \neq \xi_{\gamma}\text{-Int}(A) \cup \xi_{\gamma}\text{-Int}(B) \).

**Example 4.20** Consider \( X = \{a, b, c\} \) with discrete topology on \( X \). Define an \( \xi\text{-operation} \gamma \) on \( \xi\text{O}(X) \) by

\[
\gamma(A) = \begin{cases} 
A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \\
X & \text{otherwise}
\end{cases}
\]

Then, \( \xi_{\gamma}\text{O}(X) = \{\phi, X, \{a, b\}, \{a, c\}\} \). Now, if we let \( A = \{a, b\} \) and \( B = \{a, c\} \), then \( \xi_{\gamma}\text{-Int}(A) = \{a, b\} \) and \( \xi_{\gamma}\text{-Int}(B) = \{a, c\} \), therefore \( \xi_{\gamma}\text{-Int}(A) \cap \xi_{\gamma}\text{-Int}(B) = \{a\} \), but \( \xi_{\gamma}\text{-Int}(A \cap B) = \phi \), where \( A \cap B = \{a\} \), this implies that \( \xi_{\gamma}\text{-Int}(A) \cap \xi_{\gamma}\text{-Int}(B) = \xi_{\gamma}\text{-Int}(A \cap B) \).

The following two results can be easily proved.

**Proposition 4.21** For any subset \( A \) of a topological space \( X \), \( \xi_{\gamma}\text{-Int}(A) \subseteq \xi\text{Int}(A) \).

**Proposition 4.22** Let \( A \) be any subset of a topological space \( X \), and \( \gamma \) be an operation on \( \xi\text{O}(X) \). Then, \( \xi_{\gamma}\text{-Int}(A) = A \setminus \xi_{\gamma}\text{D}(X \setminus A) \).
REFERENCES


