Existence, Uniqueness and Stability Theorems for Certain Functional Fractional Initial Value Problem

Joseph G. Abulahad  Shayma A. Murad

College of Education  University of Dohuk, Iraq

Received on: 21 / 10 / 2009  Accepted on: 11 / 4 / 2010

ABSTRACT

In this paper, we deal with non-linear functional fractional differential equation with initial condition in $L_1$ space. We will study the existence, uniqueness and stability of the solution of fractional differential equation.

Keywords: Fractional differential equation; Riemann Liouville Fractional derivative; Stability.

1. Introduction

The fractional differential equation with initial conditions

$$y^{(\alpha)}(x) = f(x, y(x)) \quad n - 1 < \alpha < n, n \geq 2, \alpha \in R$$

$$y^{(\alpha-i)}(0) = c_i, \quad c_i \in R \quad i = 1, 2, ..., n, \quad c_n = 0$$

where $f$ is assumed to be continuous on $R \times R$, has been studied by many authors (see [1,8]). El-Salam [6] discussed a nonlinear weighted Cauchy type problem of a fractional differ-integral equation of fractional order which has the form

$$D^\alpha u(t) = f(t, u(\varphi(t))) \quad 0 < \alpha < 1$$

$$t^{1-\alpha}u(t) \bigg|_{t=0} = b$$

and proved some local and global existence, uniqueness and stability theorems for problem (1) and (2). El-Sayed [7] studied the existence theorem of $L_p$-solution of weighted Cauchy type problem of fractional differ-integral function equation and proved uniqueness and stability of the solution for equation (1) and (2).

The purpose of this work is to generalize the work of [7] to study the existence, uniqueness and stability of the solution of the initial value problem which has the form

$$y^{(\alpha)}(x) = f(x, y(\phi(x))) \quad n - 1 < \alpha < n, n \geq 2, \alpha \in R$$

with initial condition

$$y^{(\alpha-i)}(0) = c_i, \quad c_i \in R \quad i = 1, 2, ..., n, \quad c_n = 0$$
2. Preliminaries

Let \( L_1(I) \) be the class of Lebesgue integrable functions on the interval \( I=[a,b] \), where \( 0 \leq a < b < \infty \) and let \( \Gamma(.) \) be the gamma function. Recall that the operator \( T \) is compact if it is continuous and maps bounded sets into relatively compact ones. The set of all compact operators from the subspace \( U \subset X \) into the Banach space \( X \) is denoted by \( C(U,X) \).

Moreover, we set \( B_r = \{ u \in L_1(I) : \|u\| < r, r > 0 \} \) and \( \|u\| = \int_0^{\infty} e^{-u(t)}dt \).

**Definition 1:** [2]

Let \( f \) be a function which is defined almost everywhere (a.e) on \([a,b] \). for \( \alpha > 0 \), we define

\[
\int_a^b f(t) dt = \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} f(t) dt
\]

Provided that this integral exists, where \( \Gamma \) is gamma function.

**Definition 2:** [9]

For a function \( f \) defined on the interval \([a,b] \), the \( \alpha \)th Riemann-liouville fractional order derivative of \( f \) is defined by

\[
(D^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t-s)^{n-\alpha-1} f(s, y(s))ds
\]

where \( n=[\alpha]+1 \) and \([\alpha]\) denotes the integer part of \( \alpha \).

**Lemma 1:** [1]

Let \( \alpha, M > 0 \). If \( f \) is continuous and \( |f| \leq M \) for all \( x \in (a, b) \), then

\[
\lim_{x \to a^+} D^{-\alpha} f(x) = 0.
\]

**Remark 1:** [1]

Let the assumptions of the Lemma(1) be satisfied, then we define \( D^{-\alpha} f = 0 \).

**Definition 3:** [2]

If \( \alpha \in \mathbb{R} \) and \( f(x) \) is defined a.e on \( a \leq x \leq b \) we define \( f(a)(x=x) = D^{-\alpha} f \) for all \( x \in [a,b] \), provided that \( D^{-\alpha} f \) exists.

**Lemma 2:** [2]

Let \( \alpha, \beta \in \mathbb{R}, \beta > -1 \). If \( x > a \) then

\[
\int_a^x (t-a)^\beta \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} = \begin{cases} 
\frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} & \text{if } \alpha+\beta \neq \text{negative integer} \\
0 & \text{if } \alpha+\beta = \text{negative integer}
\end{cases}
\]
Lemma 3: [2]

If \( \alpha > 0 \); \( n \) is the smallest integer greater than \( \alpha \), \( f(x) \) is in \( L(a, b) \) and \( \int_a^x D^{\alpha-1} f \) exists and is absolutely continuous on \([a, b]\), then \( \int_a^x D^{\alpha-1} f = k_i \) exists for \( i = 1, 2, \ldots, n \), \( \int_a^x D^{\alpha} f \) exists a.e on \([a, b]\), is in \( L(a, b) \) and

\[
\int_a^x D^{\alpha} f = f(x) - \sum_{i=1}^{n} k_i \frac{(x-a)^{\alpha-i}}{\Gamma(\alpha-i+1)}, \text{ a.e. on } a \leq x \leq b.
\]

Furthermore, the equality holds everywhere on \((a, b]\), if, in addition, \( f(x) \) is continuous on \((a, b]\).

Lemma 4: [3]

The relation

\[
\int_a^x \int_a^x f = \int_a^x \int_a^x f
\]

holds if

(i) \( \Re(\alpha) > 0, \Re(\beta) > 0 \), and \( f(x) \) is in \( C^{(0)} \) on \([a, b]\).

(ii) \( \Re(\alpha) > 0, \Re(\beta) \leq 0 \) or \( \Re(\beta + m) > 0 \), such that \( \beta \neq -m \), and \( f(x) \) is in \( C^{(m)} \) on \([a, b]\).

(iii) \( \Re(\alpha) \leq 0, \Re(\alpha + n) > 0 \), \( \Re(\beta) > 0 \), and \( f(x) \) is in \( C^{(n)} \) on \([a, b]\).

(iv) \( \Re(\alpha) \leq 0, \Re(\beta) \leq 0 \), \( \beta \neq -m \), and \( f(x) \) is in \( C^{(m+n)} \) on \([a, b]\). When \( \beta = -m \) in (ii) and (iv), then

\[
\int_a^x \int_a^x f = \int_a^x \int_a^x f - \sum_{p=0}^{m-1} \frac{(x-a)^{\alpha-m+p}}{\Gamma(\alpha+p-m+1)} f(p)(a)
\]

Theorem 1: (Roth Fixed Point Theorem) [5]

Let \( U \) be an open and bounded subset of a Banach space \( E \), let \( T \in C(\overline{U}, E) \). Then \( T \) has a fixed point if the following condition holds \( T(\partial U) \subseteq U \).

Theorem 2: (Nonlinear Alternative of Leray-Schauder Type) [5]

Let \( U \) be an open subset of convex set \( D \) in a Banach space \( E \), assume \( 0 \in U \) and \( T \in C(\overline{U}, E) \). Then either

(A1) \( T \) has a fixed point in \( \overline{U} \), or

(A2) There exists \( \gamma \in (0,1) \) and \( x \in \partial U \) such that \( x = \gamma Tx \).

Theorem 3: (Kolmogorov Compactness Criterion) [6]

Let \( \Omega \subseteq L^p(0,1) \), \( 1 \leq p \leq \infty \). If

(i) \( \Omega \) is bounded in \( L^p(0,1) \) and

(ii) \( x_h \rightarrow x \) as \( h \rightarrow 0 \) uniformly with respect to \( x \in \Omega \), then \( \Omega \) is relatively compact in \( L^p(0,1) \), where

\[
x_h(t) = \frac{1}{h} \int_t^{t+h} x(s)ds.
\]
3. Existence of Solution

We begin this section by proving the equivalence of the initial value problems (3) and (4) with the corresponding integral equation, for proof see [1]

\[
y(x) = \sum_{i=1}^{n-1} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)} + \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-s)^{\alpha-1} f(s, y(s)) ds
\]

(5)

First, we prove that \(y(x)\) satisfies the differential equation (3) almost everywhere, by definition (1) equation (5) can be written as

\[
y(x) = \sum_{i=1}^{n-1} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)} + I_{0}^{x} f
\]

(6)

Operating both sides of equation (6) by \(I_{0}^{x-a}\), we get

\[
I_{0}^{x-a} y(x) = \sum_{i=1}^{n-1} I_{0}^{x-a} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)} + I_{0}^{x-a} I_{0}^{x} f
\]

(7)

From lemma (2), we have

\[
\sum_{i=1}^{n-1} I_{0}^{x-a} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)} = 0
\]

and by lemma (4)(iii), we have

\[
I_{0}^{x-a} I_{0}^{x} f = f(x, y(x)) \text{ a.e for each } x \in I
\]

Here equation (6), becomes

\[
I_{0}^{x} y = f(x, y(x)) \text{ a.e}
\]

Finally using definition (3), we obtain

\[
y^{(\alpha)}(x) = f(x, y(x))
\]

Next, we prove that \(y(x)\) satisfies the initial condition (4) operating by \(I_{0}^{x-a}\) on both sides of equation (5), we have

\[
I_{0}^{x-a} y(x) = \sum_{i=1}^{n-1} I_{0}^{x-a} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)} + I_{0}^{x-a} I_{0}^{x} f
\]

By lemma (2), we have

\[
\sum_{i=1}^{n-1} I_{0}^{x-a} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)} = \left\{ \begin{array}{ll}
\sum_{p=1}^{i-1} \frac{c_p x^{i-p}}{\Gamma(i-p+1)} & i = 1, 2, 3, ..., n-1 \\
\sum_{p=1}^{n-1} \frac{c_p x^{n-p}}{\Gamma(n-p+1)} & i = n
\end{array} \right.
\]

(8)

Now using (8), remark (1) and definition (3), we have
Existence, Uniqueness and Stability Theorems for ...

\[ y^{(n-1)}(0) = \begin{cases} 
  c_i & i = 1, 2, 3, \ldots, n-1 \\
  0 & i = n 
\end{cases} \]

Thus \( y(x) \) satisfies the initial condition (4). Now define the operator \( T \) as

\[ Tg(x) = \sum_{i=1}^{n-1} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i + 1)} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, g(\phi(t))) dt \quad t \in (0,1) \quad (9) \]

It is necessary to find a fixed point of the operator \( T \) and here we present the main results to prove some local and global existence theorem for equations (3)-(4) in the space \( L_1(0,1) \).

Let us state the following assumptions:

(i) \( f : (0,1) \times R \rightarrow R \) be a function with the following properties

a- for each \( x \in (0,1), f(x,.) \) is continuous .

b- for each \( y \in (0,1), f(,.y) \) is measurable.

c-There exists two function \( b(x) \) such that \[ |f(x,y)| \leq b(x) \] for each \( x \in (0,1), \ y \in R \), where \( b(x) \) is measurable and bounded.

(ii) \( \phi:(0,1) \rightarrow (0,1) \) is non-decreasing and there is a constant \( M > 0 \) such that \( \phi \geq M \) a.e on \( (0,1) \).

**Theorem 4:**

Let assumption (i) and (ii) are satisfied. If \( \sup b(x) \leq M \Gamma(1+\alpha) \). Then the initial value problems (3)-(4) has a solution \( y \in B_r \) where

\[ r \geq \sum_{i=1}^{n-1} \frac{c_i}{\Gamma(\alpha-i + 1)} \left( 1 - \frac{\sup b(x)}{M \Gamma(1+\alpha)} \right) \]

**Proof.** Let \( y \) be an arbitrary element in \( B_r, B_r = \{ u \in L_1(I) : \|u\| < r, r > 0 \} \). Then from the assumption (i) – (ii), we have

\[ \|Ty\| = \int_0^1 |Ty(x)| dx \]

from equation (9) we obtain

\[ = \int_0^1 \left| \sum_{i=1}^{n-1} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i + 1)} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, y(\phi(t))) dt \right| dx \]

\[ \leq \int_0^1 \left| \sum_{i=1}^{n-1} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i + 1)} \right| dx + \frac{1}{\Gamma(\alpha)} \int_0^1 \left| \int_0^x (x-t)^{\alpha-1} f(t, y(\phi(t))) dt \right| dx \]

\[ = \sum_{i=1}^{n-1} \frac{c_i}{\Gamma(\alpha-i + 2)} + \frac{1}{\Gamma(\alpha)} \int_0^1 \left| \int_0^x (x-t)^{\alpha-1} f(t, y(\phi(t))) dt \right| dx \]
\[
\sum_{i=0}^{n-1} \frac{c_i}{\Gamma(\alpha - i + 2)} + \frac{1}{\Gamma(\alpha)} \int_0^1 (x-t)^{\alpha-1} f(t, y(\phi(t))) dt
\]

\[
\sum_{i=0}^{n-1} \frac{c_i}{\Gamma(\alpha - i + 2)} + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha} f(t, y(\phi(t))) dt
\]

For \( t \in (0,1) \), then \((1-t)^{\alpha} < 1 \), we obtain

\[
\leq \sum_{i=0}^{n-1} \frac{c_i}{\Gamma(\alpha - i + 2)} + \frac{1}{\Gamma(\alpha + 1)} \int_0^1 f(t, y(\phi(t))) dt
\]

by using condition i(c), we obtain

\[
\leq \sum_{i=0}^{n-1} \frac{c_i}{\Gamma(\alpha - i + 2)} + \frac{1}{\Gamma(\alpha + 1)} \int_0^1 b(t) ||y(\phi(t))|| dt
\]

\[
\leq \sum_{i=0}^{n-1} \frac{c_i}{\Gamma(\alpha - i + 2)} + \frac{1}{\Gamma(\alpha + 1)} \sup_{x} b(x) \int_0^1 |y(\phi(t))| dt |
\]

\[
(10)
\]

Let \( x = \phi(t), dx = \phi'(t) dt \) and then (10) becomes

\[
\|Ty\| \leq \sum_{i=0}^{n-1} \frac{c_i}{\Gamma(\alpha - i + 2)} + \sup_{x} b(x) \int_0^1 |y(x)| dx
\]

\[
(11)
\]

The estimate (10) shows that the operator \( T \) maps the space \( L_1(0,1) \) into itself. Let \( y \in \partial B_r \), that is \( \|y\| = r \), The inequality (11) gives

\[
\|Ty\| \leq \sum_{i=0}^{n-1} \frac{c_i}{\Gamma(\alpha - i + 2)} + \frac{\sup_{x} b(x)}{M \Gamma(\alpha + 1)} r \leq r
\]

\[
(12)
\]

Which implies that

\[
\sum_{i=0}^{n-1} \frac{c_i}{\Gamma(\alpha - i + 2)} + \frac{\sup_{x} b(x)}{M \Gamma(\alpha + 1)} r \leq r
\]

\[
(13)
\]

Therefore

\[
r \geq \frac{\sum_{i=0}^{n-1} \frac{c_i}{\Gamma(\alpha - i + 1)}}{1 - \frac{\sup_{x} b(x)}{M \Gamma(1 + \alpha)}}
\]

By the hypothesis that \( \sup_{x} b(x) \leq M \Gamma(\alpha + 1) \) we conclude that \( r > 0 \). Furthermore

\[
\|f\| = \int_0^1 f(t, y(\phi(t))) dt
\]
The inequality (14) insures that \( f \in L_1(0,1) \). 

Further, since \( f \) is continuous function in \( y \) by the assumption i(a), and \( I_x^{\alpha} \) maps \( L_1(0,1) \) continuously into itself, \( I_x^{\alpha} f(t,y(\phi(t))) \) is continuous in \( y \), since \( y \) is an arbitrary element in \( B_r \), then \( T \) maps \( B_r \) continuously into \( L_1(0,1) \).

Now we claim that \( T \) is compact. Let \( \Omega \) be a bounded subset of \( B_r \), then \( (Ty)_h \to Ty \) in \( L_1(0,1) \) as \( h \to 0 \), uniformly with respect to \( Ty \in T(\Omega) \), consider

\[
\| (Ty)_h - Ty \| = \int_0^1 |(Ty)_h(x) - (Ty)(x)|dx
\]

\[
= \int_0^1 \left| \frac{1}{h} \int_x^{x+h} (Ty)(s)ds - (Ty)(x) \right|dx
\]

\[
\leq \int_0^1 \left( \frac{1}{h} \int_x^{x+h} |(Ty)(s) - (Ty)(x)|ds \right)dx
\]

\[
\leq \int_0^1 \left( \frac{1}{h} \int_x^{x+h} \sum_{i=0}^{n-1} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)} |s^{\alpha-i} - x^{\alpha-i}|ds \right. + \int_0^1 \int_0^1 |I_s^{\alpha} f(s,y(\phi(s))) - I_s^{\alpha} f(x,y(\phi(x)))|dsdx \right) \tag{15}
\]

Since \( f \in L_1(0,1) \), then \( I_s^{\alpha} f(.) \in L_1(0,1) \). Also \( \sum_{i=0}^{n-1} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)} \in L_1(0,1) \) then from theorem (3) we have

\[
\frac{1}{h} \int_x^{x+h} \sum_{i=0}^{n-1} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)} (s^{\alpha-i} - x^{\alpha-i})|ds \to 0 \text{ as } h \to 0 \text{, and}
\]

\[
\int_0^1 \int_0^1 |I_s^{\alpha} f(s,y(\phi(s))) - I_s^{\alpha} f(x,y(\phi(x)))|dsdx \to 0 \text{ as } h \to 0 \text{ almost everywhere for } x \in (0,1),
\]

from theorem (3), \( T(\Omega) \) is relatively compact and hence \( T \) is a compact operator. Let \( U = B_r \) and \( E = L_1(0,1) \), and from theorem (1), \( T \) has a fixed point. This complete the proof.

**Theorem 5:**

Let the condition (i) and (ii) of theorem (4) be satisfied.

(iii) assume that every solution \( y(x) \in L_1(0,1) \) to the equation

\[
y(x) = \sum_{i=1}^{n-1} \frac{c_i x^{\alpha-i}}{\Gamma(\alpha-i+1)} + I_0^{\alpha} f \quad \text{a.e on } (0,1), \quad n - 1 < \alpha < n, \alpha \in R, n \geq 2
\]
Satisfies \( y \neq r \) (\( r \) is arbitrary but fixed). Then the fractional initial value problem (3) and (4) has at least one solution \( y(x) \in L_1(0,1) \).

**Proof.** Let \( y(x) \) be an arbitrary element in the open set \( B_r = \{ y : \| y \| < r, r > 0 \} \). Then from (11), we have
\[
\| y \| \leq \sum_{\alpha=1}^{\alpha-1} \frac{c_\alpha}{\Gamma(\alpha-i+2)} + \sup_{M} \| b(x) \| \| y \| \quad (16)
\]
The inequality (16) implies that the operator \( T \) maps \( B_r \) continuously into \( L_1(0,1) \);
Furthermore ,from i(c),we have
\[
\| f \| \leq \sup_{M} \| b(x) \| \| y \| \quad (17)
\]
Using theorem (4) ,\( T \) maps \( B_r \) continuously into \( L_1(0,1) \) and \( T \) is compact operators.
Now set \( U = B_r \) and \( D = E = L_1(0,1) \), then by using the assumption (iii), the condition (A2) of theorem (2) does not hold. Therefore from theorem (2) \( T \) has a fixed point .This complete the proof.

**Theorem 6: (Uniqueness of The Solution)**

Let the assumption (ii) of theorem (4) be satisfied .Let the right hand side \( f(x, y) \) of the fractional differential equation satisfies the Lipschitz condition that is
\[
|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|.
\]
Then the initial value problem (3)and (4)has a unique solution.

**Proof.** Let \( y_1(x) \) and \( y_2(x) \) be any two solution of the integral equation(5), then consider
\[
y_2(x) - y_1(x) = \sum_{\alpha=1}^{\alpha-1} \frac{c_\alpha x^{\alpha-i}}{\Gamma(\alpha-i+1)} + I_0^x f(s, y_2(\phi(s))) - \sum_{\alpha=1}^{\alpha-1} \frac{c_\alpha x^{\alpha-i}}{\Gamma(\alpha-i+1)} - I_0^x f(s, y_1(\phi(s)))
\]
or \( y_2(x) - y_1(x) = I_0^x f(s, y_2(\phi(s))) - I_0^x f(s, y_1(\phi(s))) \)
\[
y_2(x) - y_1(x) = \frac{L}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} |f(s, y_2(\phi(t))) - f(s, y_1(\phi(t)))| dt
\]
\[
|y_2(x) - y_1(x)| \leq \frac{L}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} |y_2(\phi(t)) - y_1(\phi(t))| dt \quad (18)
\]
Now by using the Lipschitz condition, we have
\[
|y_2(x) - y_1(x)| \leq \frac{L}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} |y_2(\phi(t)) - y_1(\phi(t))| dt
\]
integrating both sides from 0 to 1 with respect to \( x \),we get
Existence, Uniqueness and Stability Theorems for ... 

\[
\int_0^1 |y_2(x) - y_1(x)| dx \leq \frac{L}{\Gamma(\alpha)} \int_0^1 (x-t)^{\alpha-1} |y_2(\phi(t)) - y_1(\phi(t))| dt dx
\]

or \[\|y_2(x) - y_1(x)\| \leq \frac{L}{\Gamma(\alpha)} \int_0^1 (x-t)^{\alpha-1} |y_2(\phi(t)) - y_1(\phi(t))| dt dx\]

\[
= \frac{L}{\Gamma(\alpha)} \int_0^1 (x-t)^{\alpha-1} dx |y_2(\phi(t)) - y_1(\phi(t))| dt
\]

\[
= \frac{L}{\Gamma(\alpha)} \int_0^1 \frac{(1-t)^{\alpha}}{\alpha} |y_2(\phi(t)) - y_1(\phi(t))| dt
\]

For \( t \in (0,1) \), then \((1-t)^{\alpha} < 1\), we obtain

\[
\leq \frac{L}{\Gamma(\alpha+1)} \int_0^1 |y_2(\phi(t)) - y_1(\phi(t))| dt
\]

\[
\leq \frac{L}{M \Gamma(\alpha+1)} \int_0^1 |y_2(\phi(t)) - y_1(\phi(t))| |\phi'| dt
\]

\[
\leq \frac{L}{M \Gamma(\alpha+1)} \int_{\phi(0)}^{\phi(1)} |y_2(x) - y_1(x)| dx
\]

\[
\|y_2 - y_1\| \leq \frac{L}{M \Gamma(\alpha+1)} \|y_2(x) - y_1(x)\|
\]

Choose \( M \) such that \( M \Gamma(\alpha+1) > L \) equation (19) gives

\[
\|y_2 - y_1\| < \|y_2 - y_1\|
\]

Contradiction, therefore \( y_2 = y_1 \). This complete the proof.

**Theorem 7: (Stability of The Solution)**

Suppose that the assumption of theorem (4) be satisfied, then the solution of the initial value problems (3) and (4) is uniformly stable.

**Proof.** Any solution of the fractional initial value problems (3) and (4) is given by

\[
y(x) = \sum_{i=1}^{n-1} \frac{c_i X^{\alpha_i}}{\Gamma(\alpha_i-i+1)} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, y(\phi(t))) dt
\]

Let \( \bar{y}(x) \) be any other solution of (3) and (4), that

\[
\bar{y}(x) = \sum_{i=1}^{n-1} \frac{\bar{c}_i X^{\alpha_i}}{\Gamma(\alpha_i-i+1)} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, \bar{y}(\phi(t))) dt
\]

Now \( y(x) - \bar{y}(x) = \sum_{i=1}^{n-1} \frac{(c_i - \bar{c}_i) X^{\alpha_i}}{\Gamma(\alpha_i-i+1)} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} [f(t, y(\phi(t))) - f(t, \bar{y}(\phi(t)))] dt
\]
\[
\|y(x) - \bar{y}(x)\| \leq \sum_{i=1}^{n} \frac{|c_i - \bar{c}_i|}{\Gamma(\alpha - i + 1)} + \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} \left| f(t, y(t)) - f(t, \bar{y}(t)) \right| dt
\]

Using the Lipschitz condition with Lipschitz constant \( L \), we have
\[
\|y(x) - \bar{y}(x)\| \leq \sum_{i=1}^{n} \frac{|c_i - \bar{c}_i|}{\Gamma(\alpha - i + 1)} + \frac{L}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} \left| y(\phi(t)) - \bar{y}(\phi(t)) \right| dt
\]

following the technique of [6], we have
\[
e^{-Nt} \|y(x) - \bar{y}(x)\| \leq \sum_{i=1}^{n} \frac{e^{-Nt} |c_i - \bar{c}_i|}{\Gamma(\alpha - i + 1)} + \frac{e^{-Nt} L}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} \left| y(\phi(t)) - \bar{y}(\phi(t)) \right| dt
\]

Now
\[
\int_0^1 e^{-Nt} \|y(x) - \bar{y}(x)\| \leq \sum_{i=1}^{n} \int_0^1 \frac{e^{-Nt} s^{\alpha - 1}}{\Gamma(\alpha - i + 1)} ds + \frac{L}{\Gamma(\alpha)} \int_0^1 e^{-Nt} \int_0^x (x - t)^{\alpha - 1} \left| y(\phi(t)) - \bar{y}(\phi(t)) \right| dt dx
\]

\[
\leq \sum_{i=1}^{n} \int_0^1 e^{-Nt} s^{\alpha - 1} \frac{e^{-t s^{\alpha - 1}}}{\Gamma(\alpha - i + 1)} ds + \frac{L}{\Gamma(\alpha)} \int_0^1 \int_0^x e^{-Nt} (x - t)^{\alpha - 1} \left| y(\phi(t)) - \bar{y}(\phi(t)) \right| dt dx
\]

from definition of the gamma function, we have
\[
\|y - \bar{y}\| \leq \sum_{i=1}^{n} \frac{|c_i - \bar{c}_i|}{N^{\alpha - i + 1}} + \frac{L}{\Gamma(\alpha)} \int_0^1 (x - t)^{\alpha - 1} \left| y(\phi(t)) - \bar{y}(\phi(t)) \right| dt dx
\]

\[
= \sum_{i=1}^{n} \frac{|c_i - \bar{c}_i|}{N^{\alpha - i + 1}} + \frac{L}{\Gamma(\alpha)} \int_0^1 e^{-N(t - x)} (x - t)^{\alpha - 1} dx \left| y(\phi(t)) - \bar{y}(\phi(t)) \right| dt
\]

\[
= \sum_{i=1}^{n} \frac{|c_i - \bar{c}_i|}{N^{\alpha - i + 1}} + \frac{L}{\Gamma(\alpha) N^\alpha} \int_0^1 e^{-z} z^{\alpha - 1} dz \int_0^x e^{-Nz} e^{-Nz} \left| y(\phi(t)) - \bar{y}(\phi(t)) \right| dt
\]

\[
\leq \sum_{i=1}^{n} \frac{|c_i - \bar{c}_i|}{N^{\alpha - i + 1}} + \frac{L}{N^\alpha} \int_0^1 e^{-Nz} \left| y(\phi(t)) - \bar{y}(\phi(t)) \right| dt
\]

Now Let \( z = N(x - t) \rightarrow dz = N dt \rightarrow dx = \frac{dz}{N} \).

Thus equation (21), becomes
\[
\|y - \bar{y}\| \leq \sum_{i=1}^{n} \frac{|c_i - \bar{c}_i|}{N^{\alpha - i + 1}} + \frac{L}{\Gamma(\alpha) N^\alpha} \int_0^1 e^{-z} z^{\alpha - 1} dz \int_0^x e^{-Nz} e^{-Nz} \left| y(\phi(t)) - \bar{y}(\phi(t)) \right| dt
\]

\[
= \sum_{i=1}^{n} \frac{|c_i - \bar{c}_i|}{N^{\alpha - i + 1}} + \frac{L}{\Gamma(\alpha) N^\alpha} \int_0^1 e^{-z} z^{\alpha - 1} dz \int_0^x e^{-Nz} e^{-Nz} \left| y(\phi(t)) - \bar{y}(\phi(t)) \right| dt
\]

\[
\leq \sum_{i=1}^{n} \frac{|c_i - \bar{c}_i|}{N^{\alpha - i + 1}} + \frac{L}{N^\alpha} \int_0^1 e^{-Nz} \left| y(\phi(t)) - \bar{y}(\phi(t)) \right| dt
\]
\[ \leq \sum_{i=1}^{n} \frac{1}{N^{a_i}} + \frac{L}{MN^a} \int_0^{N^a} e^{-Nt} |y(\phi(t)) - \bar{y}(\phi(t))| \, dt \frac{1}{0} |\phi'| \]

Let \( s = \phi(t) \rightarrow ds = \phi'(t) \, dt \) and \( t = \phi^{-1}(s) \), then we have

\[ \leq \sum_{i=1}^{n} \frac{1}{N^{a_i}} + \frac{L}{MN^a} \int_{\phi(0)}^{\phi(1)} e^{-N\phi^{-1}(s)} |y(s) - \bar{y}(s)| \, ds \]

\[ \leq \sum_{i=1}^{n} \frac{1}{N^{a_i}} + \frac{L}{MN^a} \int_{\phi(0)}^{\phi(1)} e^{-N\phi^{-1}(s)} |y(s) - \bar{y}(s)| \, ds , \; s < t \]

\[ \leq \sum_{i=1}^{n} \frac{1}{N^{a_i}} + \frac{L}{MN^a} \|y - \bar{y}\| , \text{ Therefore } \|y - \bar{y}\| \leq \sum_{i=1}^{n} \frac{1}{N^{a_i}} + \frac{L}{MN^a} \|y - \bar{y}\| \]

or \( \|y - \bar{y}\| \leq \left( \frac{MN^a}{MN^a - L} \right) \sum_{i=1}^{n} \frac{1}{N^{a_i}} + \frac{L}{MN^a} \|y - \bar{y}\| \)

\[ \|y - \bar{y}\| \leq \sum_{i=1}^{n} \frac{MN^a}{MN^a - LN^a} \frac{1}{N^{a_i}} + \frac{L}{MN^a} \|y - \bar{y}\| \]

Then, If \( \sum_{i=1}^{n} |c_i - c_i| < \delta(t) \), then \( \|y - \bar{y}\| < \varepsilon \)

Therefore the solution of the initial value problem is uniformly stable.

**Conclusion**

We proved the existence of the solution for certain fractional differential equation by using Roth fixed point theorem in \( L_1 \) space, then we use the Laray-Schauder theorem we proved that the fractional differential equation has at least one solution then we prove the uniqueness theorem by using the Lipschitz condition. Also we discussed the stability of the solution and we proved that the solution is uniformly stable.
REFERENCES


