Hosoya Polynomial and Wiener Index of Zero-Divisor Graph of $\mathbb{Z}_n$

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ABSTRACT

Let $R$ be a commutative ring with identity. We associate a graph $\Gamma(R)$. In this paper, we find Hosoya polynomial and Wiener index of $\Gamma(\mathbb{Z}_n)$, with $n= p^m$ or $n= p^m q$, where $p$ and $q$ are distinct prime numbers and $m$ is an integer with $m \geq 2$.

Keywords: Zero-divisor graph, commutative rings, Hosoya polynomial and Wiener index.

1. Introduction

Let $R$ be a commutative ring with identity, and let $Z(R)$ be the set of all zero-divisors in $R$, and $Z^*(R)$ is the set of all non-zero zero-divisors in it. We associate a simple graph $\Gamma(R)$ to $R$ with vertices $Z^*(R)$, and for two distinct vertices $x, y \in Z(R)$, there is an edge connecting $x$ and $y$ if and only if $xy = 0$.

The notion of a zero divisor graph of a commutative ring was first introduced in 1988 by Beck in [5], where he was interested in colorings. This investigation of coloring of a commutative ring was then continued by Anderson and  Naseer in [3], and further Anderson and Livingston in [2] associate a graph $\Gamma(R)$ to $R$. The principal ideal of an $R$ is an ideal that is generated by one element of $R$, say $a$, and usually denoted by $(a)$. The ring $R$ is called local ring if it contains exactly one maximal ideal.

A graph $G$ is said to be connected [6] if there is a path between any two distinct vertices of $G$. For vertices $x$ and $y$ of $G$, let $d(x, y)$ be the length of a shortest path from $x$ to $y$. The diameter of $G$ is defined by $\text{diam}(G) = \max \{d(x, y) : x, y \in V(G)\}$, where $V(G)$ is the set of all vertices of $G$. A graph is complete if every two of its vertices are adjacent, so the complete graph of order $n$ is denoted by $K_n$. The complement $\overline{K}_n$ of the
complete graph $K_n$ has $n$ vertices and no edges, and is referred to as the empty graph of order $n$. The subsets $V_1, V_2, \ldots, V_r$, $r \geq 2$, are called $r$-partite of the set $V(G)$, if $V_i$ is non empty, and the intersection between $V_i$ and $V_j$ is empty for any $1 \leq i, j \leq r$ with $i \neq j$, so that $\bigcup_{i=1}^{r} V_i = V(G)$.

Hosoya polynomial of the graph $G$ is defined by : $H(G ; x) = \sum_{k=0}^{\text{diam}(G)} d(G, k)x^k$, where $d(G, k)$ is the number of pairs of vertices of a graph $G$ that are at distance $k$ apart, for $k=0,1,2, \ldots, \text{diam}(G)$. The Wiener index of $G$ is defined as the sum of all distances between vertices of the graph, and denoted by $W(G)$, and we can find this index by differentiating Hosoya polynomial for the given distance with respect to $x$ and putting $x = 1$. See [7, 9].

As usual we shall assume that $p$ and $q$ are distinct positive prime numbers and $m$ be an integer with $m \geq 2$. In [1] Ahmadi and Nezhad proved some results concerning the Wiener index of $\Gamma(Z_n)$, where $n = p^2$, $pq$ and $p^2q$. In this paper we extended these results to $n = p^m$, $p^mq$.

**2. Hosoya Polynomial and Wiener Index of $\Gamma(Z_p^m)$**

In this section, we find the Hosoya polynomial and the Wiener index of $\Gamma(Z_p^m)$. It is clear that $Z^*(Z_p^m) = (p) \backslash \{0\} = \{ kp : k = 1,2,3, \ldots, p^2-1 \}$. We shall begin this section with the following lemma :

**Lemma 2.1** [8, Lemma 2.1.] Let $Z_n$ be a ring of integers modulo $n$. Then, the number of all non-zero zero-divisors for $k|n$ are $\frac{n}{k} - 1$.

**Theorem 2.2** : $\Gamma(Z_p^m) \cong K_{p-1} + \overline{K}_{p^2-p}$.

**Proof** : Since $p$ is a prime number, then it is clear that the ring $Z_p^3$ is a local ring, so we have $Z^*(Z_p^3) = (p) \backslash \{0\} = \{ kp : k = 1,2,3, \ldots, p^2-1 \}$.

Now, we can classify $Z^*(Z_p^3)$ into the two disjoint subsets as follows :

$A_1 = (p^2) \backslash \{0\}$, and $A_2 = (p) \backslash \{A_1 \cup \{0\}\}$. It is clear that $Z^*(Z_p^3) = A_1 \cup A_2$ and by using Lemma 2.1 we have $|A_1| = p^3 - 1 = p - 1$, and $|A_2| = \frac{p^3}{p} - (\frac{p^3}{p^2}+1) = p^2-p$, so we can write $A_1 = \{ kp^2 : k = 1,2,3, \ldots, p^2-1 \}$ and $A_2 = \{ kp : k = 1,2,3, \ldots, p^2-1 : p \nmid k \}$. Now, let $x,y \in Z^*(Z_p^3)$. Then, there are three cases :

**Case 1** : If $x,y \in A_1$, then there exists positive integers $k_1$ and $k_2$ with $p \nmid k_1,k_2$ such that $x = k_1p^2$ and $y = k_2p^2$, and we have $xy = k_1p^2k_2p^2 = k_1k_2p^4 \equiv 0 \text{ mod } p^3$, then $x$ adjacent with $y$ in this case.

**Case 2** : If $x \in A_1$ and $y \in A_2$, then there exists positive integers $k_1$ and $k_2$ with $p \nmid k_1,k_2$ such that $x = k_1p^2$, and $y = k_2p$, and we have $xy = k_1p^2k_2p = k_1k_2p^3 \equiv 0 \text{ mod } p^3$, then $x$ adjacent with $y$ in this case.

**Case 3** : If $x,y \in A_2$, then there exists positive integers $k_1$ and $k_2$ with $p \nmid k_1,k_2$ such that $x = k_1p$ and $y = k_2p$, and we have $xy = k_1p k_2p = k_1k_2p^2 \equiv 0 \text{ mod } p^3$, then $x$ and $y$ are not adjacent in this case.

From the previous, we see that every vertex in $A_1$ is adjacent with any other vertex in $A_1$ and $A_2$, so that no vertex in $A_2$ is adjacent with any other vertex in $A_2$, therefore we have : $\Gamma(Z_p^3) \cong K_{|A_1|} + \overline{K}_{|A_2|} = K_{p-1} + \overline{K}_{p^2-p}$.

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Theorem 2.3: \( H(\Gamma(Z_{p^3})), x) = a_0 + a_1 x + a_2 x^2 \), where \( a_0 = p^2 - 1 \), \( a_1 = \frac{1}{2} (2p^3 - 3p^2 - p + 2) \), and \( a_2 = \frac{1}{2} (p^4 - 2p^3 + p) \).

**Proof**: From clearly that diam(\( \Gamma(Z_{p^3}) \))= d(x,y)= 2, for all x,y \( \in A_2 \), therefore \( H(\Gamma(Z_{p^3})), x) = a_0 + a_1 x + a_2 x^2 \), where \( a_i = d(\Gamma(Z_{p^3})), i \) for \( i = 0,1,2 \). It is clear that \( a_0 = d(\Gamma(Z_{p^3})), 0 \) = \( |Z'(Z_{p^3})| = p^2 - 1 \).

Now, let \( Z'(Z_{p^3}) = A_1 \cup A_2 \), where \( A_1 = (p^3 \setminus \{0\}) \) and \( A_2 = (p \setminus \{A_1 \cup \{0\} \}) \) and by Lemma 2.1 we have, \( |A_1| = p - 1 \), and \( |A_2| = p^2 - p \).

To find \( a_1 \), let x,y \( \in Z'(Z_{p^3}) \) such that d(x,y)= 1, from the proof of Theorem 2.2 we get that d(x,y)= 1 if and only if x,y \( \in A_1 \) or x \( \in A_1 \) and y \( \in A_2 \), then we have:

\[
a_1 = d(\Gamma(Z_{p^3})), 1) = \binom{p^2}{2} + \binom{p - 1}{2} = \frac{1}{2} (2p^2 - 3p^2 - p + 2).
\]

To find \( a_2 \), let x,y \( \in Z'(Z_{p^3}) \) such that d(x,y)= 2, from the proof of Theorem 2.2, we have d(x,y)= 2 if and only if x,y \( \in A_2 \), then we have:

\[
a_2 = d(\Gamma(Z_{p^3})), 2) = \binom{p - 1}{2} + \frac{1}{2} (2p^4 - 2p^3 + p + 2).
\]

**Corollary 2.4**: \( W(\Gamma(Z_{p^3})) = \frac{1}{2} (2p^2 - 2p^3 - 3p^2 + p + 2) \).

**Proof**: Since \( W(\Gamma(Z_{p^3})), x) = d(x, \Gamma(Z_{p^3})), x|_{x=1}, \) then we have \( W(\Gamma(Z_{p^3})) = 0 + \frac{1}{2} (2p^2 - 3p^2 - p + 2) + 2x \binom{1}{2} (p^4 - 2p^3 + p) \)|\( x=1 \)

\[
= \frac{1}{2} (2p^2 - 2p^3 - 3p^2 + p + 2).
\]

Next, we give the following definition.

**Definition 2.5**: Let \( Z_{p^m} \) be the ring of integers modulo \( p^m \). Then we can write \( Z'(Z_{p^m}) = \bigcup_{i=1}^{m-1} A_i \), where \( A_i \) are disjoint subsets of \( Z'(Z_{p^m}) \), for \( 1 \leq i \leq m-1 \), which are defined as follows:

\[
A_1 = \{p^m - 1\}, A_2 = \{p^{m-2}\} \setminus \{A_1 \cup \{0\} \}, A_3 = \{p^{m-3}\} \setminus \{A_1 \cup A_2 \cup \{0\} \}, \ldots,
\]

\[
A_{m-1} = \{p\} \setminus \{ \bigcup_{i=1}^{m-2} A_i \cup \{0\} \}.
\]

Notice that, from Lemma 2.1, we get

\[
|A_i| = p^i - p^{i-1}, \text{ for any } 1 \leq i \leq m - 1, \text{ so that we can write}
\]

\[
A_i = \{k_i \mid p^i - p^{i-1} \leq k_i \leq p^i \}, \text{ for any } 1 \leq i \leq m - 1.
\]

**Lemma 2.6**: Let \( A_i \), for \( 1 \leq i \leq m - 1 \) be subsets of \( Z'(Z_{p^m}) \) which are defined in Definition 2.5 and let s and t be two integers with \( 1 \leq s \leq t \leq m - 1 \), then \( \sum_{i=s}^{t} A_i \mid = p^{t-s} p^{t-1} \).

**Proof**: Since, \( |A_i| = p^i - p^{i-1}, \forall 1 \leq i \leq m - 1 \), then we have

\[
\sum_{i=s}^{t} A_i \mid = \sum_{i=s}^{t} (p^i - p^{i-1}) = p^s - p^{t-1} + p^{t+1} - p^s + \ldots + p^{t+1} - p^s - p^t - p^{t-1} = p^{t-s} p^{t-1}.
\]

**Theorem 2.7**: Let \( A_i \), for \( 1 \leq i \leq m - 1 \), be subsets of \( Z'(Z_{p^m}) \) which are defined in Definition 2.5. Then, for any x,y \( \in Z'(Z_{p^m}) \), xy = 0 if and only if x \( \in A_i \) and y \( \in A_j \) such that i \( \leq j \), for some m ≤ i, j ≤ m - 1.

**Proof**: From Definition 2.5 we have \( Z'(Z_{p^m}) = \bigcup_{i=1}^{m-1} A_i \), where \( A_i = \{k_i \mid p^i - p^{i-1} \leq k_i \leq p^i \} \), for \( 1 \leq i \leq m - 1 \). Now, for any \( 1 \leq i, j \leq m - 1 \), let x \( \in A_i \) and y \( \in A_j \). Then, there exists two positive integers k_i and k_j such that x \( \equiv k_i p^{m-i} \) and y \( \equiv k_j p^{m-j} \), with \( p \nmid k_i k_j \).

Now, if xy = 0. Then, xy = k_i p^{m-i} k_j p^{m-j} = k_i k_j p^{2m-(i+j)} \equiv 0 \mod p^m \). And since \( k_i k_j \equiv 0 \mod p^m \), therefore \( p^{2m-(i+j)} \equiv 0 \mod p^m \), and that means \( p^m \) divides \( p^{2m-(i+j)} \), which implies that \( 2m-(i+j) \geq m \), therefore i + j ≤ m.
Conversely: Let $x \in A_i$ and $y \in A_j$ such that $i + j \leq m$ for some $1 \leq i, j \leq m-1$, and suppose contrary that $xy \neq 0 \Rightarrow xy = k_i k_j p^{2m-(i+j)} \not\equiv 0 \pmod{p^m}$, and since, $p \nmid k_i k_j$, therefore $p^m \nmid p^{2m-(i+j)}$. Then, we get $2m-(i+j) < m$, so that $2m - m < i+j$, which implies that $i+j > m$, this contradiction, therefore $xy = 0$.

From Theorem 2.7 and Lemma 2.6 we can give the general form of the graph $\Gamma(Z_{p^t})$, where $t=4,5$, as the following:

![Diagram](image1)

**Figure 2.1**: The general form of the graph $\Gamma(Z_{p^4}) \cong K_{(p-1)} + (K_{(p^2-p)} \cup \overline{K}_{(p^3-p^2)})$

![Diagram](image2)

**Figure 2.2**: The general form of the graph $\Gamma(Z_{p^4})$

![Diagram](image3)

**Figure 2.3**: The general form of the graph $\Gamma(Z_{p^5})$

We can now give the general form of the graph $\Gamma(Z_{p^m})$:
Theorem 2.8: The graph \( \Gamma(Z_{p^m}) \) is \( s \)-partite graph, where

\[
s = \begin{cases} 
  \frac{p^{m-1}}{2} & \text{if } m \text{ is an odd number} \\
  \frac{p^{m-1}}{2} - 1 & \text{if } m \text{ is an even number}
\end{cases}
\]

Proof: From Definition 2.5, we have \( Z^*(Z_{p^m}) = \bigcup_{i=1}^{m-1} A_i \), where \( A_i = \{k_ip^m, k_i = 1, 2, \ldots, p^i - 1 : p \nmid k_i \} \), for \( 1 \leq i \leq m-1 \).

Suppose that \( m \) is an odd number, we see that by Theorem 2.7, any two distinct vertices lie in \( \bigcup_{i=1}^{m} A_i \) are adjacent because that \( i + j \leq m \), for any \( 1 \leq i, j \leq \frac{m-1}{2} \), this means that, we cannot put the vertices of the sets \( A_1, A_2, \ldots, A_{m-1} \) in less than \( \sum_{i=1}^{m-1} |A_i| = p^{m-1} \frac{m-1}{2} - 1 \) of partite sets. also by Theorem 2.7 we see that any vertex \( x \in A_{m+1} \) is adjacent with every vertex of \( \bigcup_{i=1}^{\frac{m-1}{2}} A_i \) because that \( \frac{m+1}{2} + i \leq m \), for any \( 1 \leq i \leq \frac{m-1}{2} \), so that \( x \) is not adjacent with any other vertex in \( A_{m+1} \) because that \( 2(\frac{m+1}{2}) > m \), therefore we must consider new partite set, say \( V \), contains the vertices of \( A_{m+1} \), in this case, we cannot put the vertices of the sets \( A_1, A_2, \ldots, A_{m+1} \), in less than \( (p^{m-1} \frac{m-1}{2} - 1) \).
1) + 1 = p^{m-1}/z \) of partite sets. Now, if we can put the vertices of \( \bigcup_{i=k}^{m-1} A_i \) in V, then the theorem hold, that is: by Theorem 2.7 we see that any two distinct vertices in \( \bigcup_{i=k}^{m+3} A_i \) are not adjacent because that \( i+j > m \) for any \( \frac{m+3}{2} \leq i, j \leq m - 1 \), so that any vertex in V is not adjacent with every vertex of \( \bigcup_{i=k}^{m+3} A_i \) because that \( \frac{m+1}{2} + i > m \), for any \( \frac{m+3}{2} \leq i \leq m - 1 \), and this shows that we cannot put the vertices of \( Z^*(Z_p^m) = \bigcup_{i=1}^{m-1} A_i \) in less than \( p^{m-1}/z \) of partite sets, therefore \( \Gamma(Z_p^m) \) is \( p^{m-1}/z \)-partite graph.

Now, let m be an even integer number, similarly we cannot put the vertices of the set \( \bigcup_{i=1}^{m/2} A_i \) in less than \( \sum_{i=1}^{m/2} |A_i| = p^{m/2} - 1 \) of partite sets, say \( V_1, V_2, \ldots, V_{m/p-1} \), each of these partite sets contains only one vertex of the set \( \bigcup_{i=1}^{m/2} A_i \), suppose that the partite set \( V_m/p-1 \) contains one of the vertices of the set \( A_m \), and we are going to show that we can put the vertices of the set \( \bigcup_{i=m/2}^{m-1} A_i \) in the partite set \( V_m/p-1 \), that is: by Theorem 2.7 we see that any two distinct vertices in the set \( \bigcup_{i=m/2}^{m-1} A_i \) are not adjacent because that \( i+j > m \) for any \( \frac{m+2}{2} \leq i, j \leq m - 1 \), so that any vertex of the set \( \bigcup_{i=m/2}^{m+1} A_i \) is not adjacent with every vertex of the set \( A_m \) because that \( \frac{m}{2} + i > n \) for any \( \frac{m+2}{2} \leq i \leq m - 1 \), and this shows we can put the vertices of the set \( \bigcup_{i=m/2}^{m-1} A_i \) in the partite set \( V_m/p-1 \), therefore we cannot put the vertices of \( Z^*(Z_p^m) = \bigcup_{i=1}^{m-1} A_i \) in less than \( p^{m/2} - 1 \) of partite sets, hence \( \Gamma(Z_p^m) \) is \( (p^{m/2} - 1) \)-partite graph. □

**Lemma 2.9** [7]: Let G be a connected graph of order r. Then
\[
\sum_{i=0}^{diam(G)} d(G, i) = \frac{1}{2} r (r+1).
\]
Now, we give the main result in this section.

**Theorem 2.10**: \( H(\Gamma(Z_p^m)); \ x) = a_0 + a_1 x + a_2 x^2 \), where
\[
a_0 = p^{m-1} - 1,
\]
\[
a_1 = \frac{1}{2} \left( m - 1 \right) p^m - m p^{m-1} - p\left[\frac{m}{2}\right] + 2, \text{ and }
\]
\[
a_2 = \frac{1}{2} \left[ p^{2(m-1)} - (m-1) p^m + (m-3) p^{m-1} + p\left[\frac{m}{2}\right] \right].
\]

**Proof**: When \( m = 2 \), we have \( \Gamma(Z_p^2) \cong K_{p-1} \), and the theorem is true in this case.

Now, suppose that \( m \geq 3 \), since \( Z_p^m \) is a local ring, then by [4, Theorem 2.3.], there is a vertex adjacent with every other vertices in \( \Gamma(Z_p^m) \), this means that \( diam(\Gamma(Z_p^m)) = 2 \), therefore \( H(\Gamma(Z_p^m)); \ x) = a_0 + a_1 x + a_2 x^2 \), where \( a_i = d(\Gamma(Z_p^m), i) \), for \( i = 0, 1, 2 \).

To find \( a_0 \), by Lemma 2.1 we have
\[
a_0 = d(\Gamma(Z_p^m), 0) = | Z^*(Z_p^m) | = \frac{p^m}{p} - 1 = p^{m-1} - 1.
\]
To find \( a_1 \), suppose that \( m \) be an odd number, and let \( x, y \in Z^*(Z_p^m) \), since \( Z^*(Z_p^m) = \bigcup_{i=1}^{m-1} A_i \), then by Theorem 2.7 we see that \( d(x, y) = 1 \) (i.e. \( xy = 0 \)) if and only if
x \in A_i \text{ and } y \in A_j \text{ such that } i + j \leq m, \text{ for some } 1 \leq i, j \leq m-1, \text{ and this holds if and only if one of the following two cases holds:}

**Case 1**: $1 \leq i, j \leq \frac{m-1}{2}$, because that $i+j \leq m$ for any $1 \leq i, j \leq \frac{m-1}{2}$, in this case there are $m_1$ edges where

$$m_1 = \left( \sum_{i=1}^{\frac{m-1}{2}} |A_i| \right) \left( \frac{m-1}{2} - 1 \right) = \frac{1}{2} (\frac{m-1}{2} - 1) \left( \frac{m-1}{2} - 2 \right) \ldots (\ast).$$

**Case 2**: $1 \leq i \leq \frac{m-1}{2}$ and $\frac{m+1}{2} \leq j \leq m - i$, since that $i+j \leq m$ for any $1 \leq i \leq \frac{m-1}{2}$ and $\frac{m+1}{2} \leq j \leq m - i$, in this case there are $m_2$ edges where

$$m_2 = \sum_{i=1}^{\frac{m-1}{2}} (|A_i| \sum_{j=\frac{m+1}{2}}^{m-1} |A_j|), \text{ since } |A_i| = p^i - p^{i-1}, \text{ for each } 1 \leq i \leq m-1, \text{ and by using Lemma 2.6 we get:}$$

$$m_2 = \sum_{i=1}^{\frac{m-1}{2}} (p^i - p^{i-1})(p^{m-i} - p^{\frac{m-1}{2}}) = \sum_{i=1}^{\frac{m-1}{2}} (p^i - p^{i-1})(p^{m-1} - p^{\frac{m-3}{2}})$$

$$= \sum_{i=1}^{\frac{m-1}{2}} (p^i - p^{i-1})(p-1)(p^{m-1} - p^{\frac{m-3}{2}}) = \sum_{i=1}^{\frac{m-1}{2}} (p^i - p^{\frac{m-3}{2}}) \sum_{i=1}^{\frac{m-1}{2}} p^i$$

$$= \frac{m-1}{2} (p^m - p^{\frac{m+1}{2}}) - \frac{m-1}{2} \left( \frac{m-1}{2} - 1 \right) \ldots (\ast\ast).$$

Now, from (\ast) and (\ast\ast), we get

$$a_1 = m_1 + m_2 = \frac{1}{2} \left( p^m - 1 \right) \left( p^{\frac{m-1}{2}} - 1 \right) + \frac{1}{2} \left( p^m - 1 \right) \left( p^{\frac{m-1}{2}} - 1 \right)$$

$$= \frac{1}{2} [(m-1)p^m - m p^{m-1} - p^{\frac{m+1}{2}} + 2].$$

Similarly, when an $m$ be an even number we get that

$$a_1 = \frac{1}{2}.$$

$$[(m-1)p^m - m p^{m-1} - p^{\frac{m+1}{2}} + 2].$$

Hence $a_1 = \frac{1}{2} [(m-1)p^m - m p^{m-1} - p^{\frac{m+1}{2}} + 2].$

Next, to find $a_2$ we shall use lemma 2.9, and we get:

$$a_2 = \frac{1}{2} [a_0 (a_0 + 1) - a_0 - a_1]$$

$$= \frac{1}{2} \left[ (m-1)p^m - (m-1)p^{m-1} - \frac{1}{2} [(m-1)p^m - m p^{m-1} - p^{\frac{m+1}{2}} + 2] \right]$$

$$= \frac{1}{2} \left[ p^{(m-1)} - (m-1)p^m - (m-3)p^{m-1} + p^{\frac{m+1}{2}} \right].$$

**Corollary 2.11**: $W(G(Z_{p^m})) = \frac{1}{2} \left[ 2p^{(m-1)} - (m-1)p^m + (m-6)p^{m-1} + p^{\frac{m+1}{2}} \right].$

3. **Hosoya Polynomial and Wiener Index of $\Gamma(Z_{p^m})$.**

In this section, we find the Hosoya polynomial and the Wiener index of $\Gamma(Z_{p^m})$. First, we shall give the following lemma:
Lemma 3.1: The number of all non-zero zero-divisors of a ring \( Z_{p^m q} \) is 
\((p+q-1)p^{m-1}-1\).

Proof: Since, \( p \) and \( q \) are distinct prime numbers, then clearly 
\( Z(R) = (p)U(q) \), therefore \( Z^*(R) = \{(p)(q)\} - \{0\} \).

Now, let \( x \in Z^*(R) \), then either \( x \in (p) \) or \( x \in (q) \) with \( x \notin (pq) \), so by Lemma 2.1 we get :
\[
|Z^*(R)| = \left(\frac{p^{mq}}{q} - 1\right) + \left(\frac{p^{mq}}{p} - 1\right) - \left(\frac{p^{mq}}{pq} - 1\right)
= (p^{m-1}q - 1) + (p^m - 1) - (p^{m-1} - 1)
= p^{m-1}q - 1 + p^m - 1 - p^{m-1} + 1
= (p+q-1) p^{m-1} - 1. \]

Definition 3.2: Let \( Z_{p^m q} \) be the ring of integers modulo \( p^m q \), then we can write :
\( Z^*(Z_{p^m q}) = \bigcup_{i=1}^{m} (B_i U C_i) \), where \( B_i \) and \( C_i \), are disjoint subsets of \( Z^*(Z_{p^m q}) \), for \( 1 \leq i \leq m \), which are defined as follows :
\[ B_i = (p^{m-1}q - 1) \bar{B}_i \{0\}, B_2 = (p^{m+2}q) \{0\}, \ldots, B_m = (q) \{0\}, \]
\[ C_i := \{0\}, C_2 := (p^{m-1}) \{0\}, \ldots, C_m := (p) \{0\}. \]

Notice that, by Lemma 2.1 we get :
\[ |B_i| = p^i - p^{i-1}, \text{ for any } 1 \leq i \leq m, \quad |C_i| = (q-1) \text{ and } |C_i| = (p^i - p^{i-1})(q-1) \text{, for all } 2 \leq i \leq m, \text{ also we can write :}
B_i = \{ kp^{m-i}q : k \in \mathbb{Z}, 1 \leq k \leq p^{m-i} - 1 \}, \quad \text{and} \quad C_i = \{ kp^{m+i}q : k \in \mathbb{Z}, 1 \leq k \leq p^{m+i}q - 1 \}. \]

Remarks:
1. \( \sum_{i=1}^{m} |B_i| = p^m - 1 \).
2. \( \sum_{i=1}^{m} |C_i| = p^{m-1}(q-1) \).
3. \( |C_i| = (q-1) \bar{B}_{i-1} \), for any \( 2 \leq i \leq m \).
4. \( |A_i| = |B_i| \), for any \( 1 \leq i \leq m-1 \), where \( A_i \), for all \( 1 \leq i \leq m-1 \), be subsets of \( Z^*(Z_{p^m}) \) which are defined in Definition 2.5.

Lemma 3.3: Let \( B_i \) and \( C_i \), for all \( 1 \leq i \leq m \), be subsets of \( Z^*(Z_{p^m q}) \) which are defined in Definition 3.2 then :

1. If \( s \) and \( t \) are two integers with \( 1 \leq s, t \leq m \), then \( \sum_{i=s}^{t} |B_i| = p^t - p^{s-1} \).
2. If \( t \) be an integer with \( 1 \leq t \leq m \), then \( \sum_{i=1}^{t} |C_i| = (q-1)p^{t-1} \).
3. If \( s \) and \( t \) are two integers with \( 2 \leq s, t \leq m \), then \( \sum_{i=s}^{t} |C_i| = (q-1)(p^t - p^{s-1}) \).

Proof: By the same method of a proof of Lemma 2.6.

Theorem 3.4: Let \( B_i \) and \( C_i \), for \( 1 \leq i \leq m \), be subsets of \( Z^*(Z_{p^m q}) \) which are defined in Definition 3.2, and let \( x, y \in Z^*(Z_{p^m q}) \). Then, \( xy = 0 \) if and only if either \( x \in B_i \) and \( y \in B_j \) with \( i+j \leq m \), or \( x \in B_i \) and \( y \in C_j \) with \( i+j \leq m+1 \), for some \( 1 \leq i, j \leq m \).

Proof: From the Definition 3.2, we have \( Z^*(Z_{p^m q}) = \bigcup_{i=1}^{m} (B_i U C_i) \). Now, let \( x, y \in Z^*(Z_{p^m q}) \) such that \( xy = 0 \), since \( x, y \in \bigcup_{i=1}^{m} (B_i U C_i) \), then there are two cases :
Case 1: \( x \in B_i \) and \( y \in B_j \) for some \( 1 \leq i, j \leq m \), in this case, there are positive integers \( k \) and \( j \) with \( i+j \leq m \), and \( i+j \leq m+1 \), for some \( 1 \leq i, j \leq m \), since \( xy = 0 \) by hypothesis, then we get \( xy = (k_1 j_1)(p^{2m+i+j})q^2 \equiv 0 \) (mod \( p^m q \)), since \( p \mid k_1, j \),
Therefore $p^{2m-(i+j)}q^2 \equiv 0 \pmod{p^mq}$, this means that $p^{2m-(i+j)}$ is divisible by $p^m$. Therefore $2m-(i+j) \geq m$, hence $i+j \leq m$.

Case 2: $x \in B_i$ and $y \in C_j$ for some $1 \leq i, j \leq m$, in this case, there are positive integers $k_i$ and $k_j$ with $p|k_i$ and $q|k_j$, such that $x= k_ip^{m-i}q$ and $y= k_jp^{m-j+1}$, for $1 \leq i, j \leq m$.

Since $xy=0$ by hypothesis, then $xy= (k_i k_j)p^{2m-(i+j)+1}q^2 \equiv 0 \pmod{p^mq}$, since $p|k_i$, $k_j$ and $q$ divides $q^2$ then $p^{2m-(i+j)+1}$ is not divisible by $p^m$, therefore $2m-(i+j)+1 < m$ which contradicts the theorem.

Conversely: Let $x \in B_i$ and $y \in B_j$ for some $1 \leq i, j \leq m$, such that $i+j \leq m$, and suppose contrary that $xy \neq 0$, we get $xy= (k_i k_j)p^{2m-(i+j)+1}q^2 \not\equiv 0 \pmod{p^mq}$, since $p|k_i$, $k_j$ and $q$ divides $q$ then $p^{2m-(i+j)+1}$ is not divisible by $p^m$, therefore $2m-(i+j)+1 < m$ which contradicts the theorem.

Finally, we see that when $x \in C_i$ and $y \in C_j$, then $xy \neq 0$ for any $1 \leq i, j \leq m$.

From previous, we get that if $xy=0$, then either $x \in B_i$ and $y \in B_j$ with $i+j \leq m$, or $x \in B_i$ and $y \in C_j$ with $i+j \leq m+1$, for some $1 \leq i, j \leq m$.

Conversely : Let $x \in B_i$ and $y \in B_j$ for some $1 \leq i, j \leq m$, such that $i+j \leq m$, and suppose contrary that $xy \neq 0$, we get $xy= (k_i k_j)p^{2m-(i+j)+1}q^2 \not\equiv 0 \pmod{p^mq}$, since $p|k_i$, $k_j$ and $q$ divides $q$ then $p^{2m-(i+j)+1}$ is not divisible by $p^m$, therefore $2m-(i+j)+1 < m$ which contradicts the theorem.

Now, let $x \in B_i$ and $y \in C_j$ for some $1 \leq i, j \leq m$, such that $i+j \leq m+1$, and suppose contrary that $xy \neq 0$, we get $xy= (k_i k_j)p^{2m-(i+j)+1}q^2 \not\equiv 0 \pmod{p^mq}$, since $p|k_i$, $k_j$ and $q$ divides $q$ then $p^{2m-(i+j)+1}$ is not divisible by $p^m$, therefore $2m-(i+j)+1 < m$ which contradicts the theorem.

Finally, we see that when $x \in C_i$ and $y \in C_j$, then $xy \neq 0$ for any $1 \leq i, j \leq m$.

From Theorem 3.4 and Lemma 3.3, we can give the general form of the graph $\Gamma(Z_{p^{t}q})$, where $t=3,4$ as follows:
We can now give the general form of the graph $\Gamma(Z_{p^m q})$, as the following:
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3.3 : The general form of the graph $\Gamma(\mathbb{Z}_{p^m q})$, where $m$ is an odd number with $m \geq 5$.

Figure 3.4 : The general form of the graph $\Gamma(\mathbb{Z}_{p^m q})$, where $m$ is an even number with $m \geq 6$.

**Lemma 3.5** [8, Proposition 3.2.] : Let $\mathbb{Z}_{p^m q}$ be a ring of integers modulo $p^m q$. Then, \( \text{diam}(\Gamma(\mathbb{Z}_{p^m q})) = 3 \).

Now, we give the main result in this section.

**Theorem 3.6** : \( H(\Gamma(\mathbb{Z}_{p^m q}); x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \), where

\[
\begin{align*}
a_0 &= (p+q-1) p^{m-1} - 1, \\
a_1 &= \frac{1}{2} \left( 2mq (p-1) - (m+1) p + m \right) p^{m-1} - \frac{1}{2} p\left\lfloor \frac{m}{2} \right\rfloor + 1, \\
a_2 &= \frac{1}{2} \left( p^2 + q^2 - 1 \right) p^{2m-2} + \frac{1}{2} \left( (m-4) p - 2(m-1) pq + (2m-5)q - m + 5 \right) p^{m-1} + \frac{1}{2} p\left\lfloor \frac{m}{2} \right\rfloor, \\
a_3 &= (q-1)(p-1) \left( p^{2m-2} - p^{m-1} \right).
\end{align*}
\]

**Proof** : By Lemma 3.5 we have \( \text{diam}(\Gamma(\mathbb{Z}_{p^m q})) = 3 \), then \( H(\Gamma(\mathbb{Z}_{p^m q}); x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \), where \( a_i = d(\Gamma(\mathbb{Z}_{p^m q}), i) \), for \( i = 0, 1, 2, 3 \).

To find $a_0$, by Lemma 3.3 we have

\[
a_0 = d(\Gamma(\mathbb{Z}_{p^m q}), 0) = |Z^*(\mathbb{Z}_{p^m q})| = (p+q-1) p^{m-1} - 1.
\]

Now, to find $a_1$, let $x, y \in Z^*(\mathbb{Z}_{p^m q})$ such that $d(x, y) = 1$ (i.e. $xy = 0$), hence by using Theorem 3.4 there are two cases:
Case 1: \(x \in B_i\) and \(y \in B_j\) with \(i+j \leq m\), for some \(1 \leq i,j \leq m\), the same as the proof of Theorem 2.7, we get that there are \(m_1\) edges in this case, where

\[
m_1 = \frac{1}{2} [(m-1) p^m - m p^{m-1} - \frac{m(m-1)}{2} + 2] \ldots (*)
\]

Case 2: \(x \in B_i\) and \(y \in C_j\) with \(i+j \leq m+1\), for some \(1 \leq i,j \leq m\), this holds if and only if \(1 \leq i \leq m\) and \(1 \leq j \leq m-i+1\), because that \(i+j \leq m+1\) for any \(1 \leq i \leq m\) and \(1 \leq j \leq m-i+1\), so that \(i+j > m+1\) in otherwise of this case, so that there are \(m_2\) edges, where

\[
m_2 = \sum_{i=1}^{m} (p^i - p^{i-1}) p^m - i \sum_{j=1}^{m} (p^{i-1} - p^{i-2}) (q-1) = \sum_{i=1}^{m} (p^{i-1} - p^{i-2}) (q-1) (p^m - i) \ldots (**)
\]

Now, from \((*)\) and \((***)\), we get that

\[
a_1 = m_1 + m_2 = \frac{1}{2} [(m-1) p^m - \frac{1}{2} m p^{m-1} - \frac{1}{2} p^{\lceil \frac{m}{2} \rceil} + 1 + m (p^m - p^{m-1})] (q-1)
\]

\[
= \frac{1}{2} m p^m - \frac{1}{2} p^{\lceil \frac{m}{2} \rceil} + 1 + m p^{m-1} q - m p^m + p^{m+1} q + mp^{m-1}
\]

Now, to find \(a_3\), for \(i=2,3\), in the first, we shall find \(a_3\).

Let \(x,y \in Z^{*}(\mathbb{Z}_p^n)\) such that \(d(x,y)=3\), then \(x \in B_i\) and \(y \in C_j\) for some \(1 \leq i,j \leq m\). In this case, see that \(d(x,y)=3\) if and only if \(i=m\) and \(2 \leq j \leq m\), because that \(d(x,y)\leq 2\) for any \(1 \leq i \leq m-1\) and \(2 \leq j \leq m\), also that \(d(x,y)=1\) for \(1 \leq i \leq m\) and \(j=1\), therefore the number of pairs of vertices that are distance three apart is \((B_m \sum_{j=2}^{m} |C_j|)\), i.e.

\[
a_3 = |B_m| \sum_{j=2}^{m} |C_j|, \text{ since } |B_m| = (p^m - p^{m-1}), \text{ then by Lemma 3.3, we get that:}
\]

\[
a_3 = (p^m - p^{m-1}) (q-1) (p^{m-1} - 1) = (q-1) (p-1) (p^{2m-2} - p^{m-1}) \ldots (**)
\]

Now, to find \(a_2\) we shall use Lemma 2.9, that is:

\[
a_2 = \frac{1}{2} a_0 (a_0+1) - a_0 - a_1 - a_3 = \frac{1}{2} a_0 (a_0-1) - a_1 - a_3
\]

\[
= \frac{1}{2} ((p+q-1) p^{m-1} - 1) ((p+q-1) p^{m-1} - 2) - \frac{1}{2} (2mq (p-1) - (m+1) p + m) p^m
\]

\[
= \frac{1}{2} p^m (p^2 + q^2 - 1) - p^{2m-1} + \frac{1}{2} [(m-4) p - 2(m-1) q + (2m-5) q - m + 5] p^{m-1} + \frac{1}{2} p^{\lceil \frac{m}{2} \rceil} .
\]

**Corollary 3.7**: \(W(G(\mathbb{Z}_p^nq)) = [p^2 + q^2 + 3(2p-q) + 2] p^{2m-2} + \frac{1}{2} [(m-3) p - 2(m+1) q + 2(m-2) q] p^{m-1} + \frac{1}{2} p^{\lceil \frac{m}{2} \rceil} + 1 .
\]

**REFERENCES**


Hosoya Polynomial and Wiener Index of Zero-Divisor Graph of \( Z_n \)


