The Graph of Annihilating Ideals

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ABSTRACT

Let R be a commutative ring with identity and AG(R) be the set of ideals with non-zero annihilators. The annihilating ideal graph AG(R) is a graph of vertex set AG (R)\{(0)} and two distinct ideal vertices I and J are adjacent if and only if I J = (0). In this paper , we establish a new fundamental properties of AG(R) as well as its connection with Γ(R).

Keywords: Annihilating ideal graph, zero divisor graph, reduced rings, finite local rings, rings integer modulo n

بيان للمثاليات التالفة

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الملخص

لتكن R حلقة إبدالية تحتوي على العنصر المحايد، وان AG(R) مجموعة المثاليات ذات تالف غير صفري. يعرف بيان المثاليات التالفة AG(R) على انه البيان الذي رؤوسه في AG(R)\{(0)} ، وان أي رأسين مثاليين مختلفين I وJ متجاورين اذا وفقط اذا IJ=(0) . في هذا البحث درسنا هذا النوع من البيانات وأعطينا العديد من شروطه الأساسية، اضافة الى ذلك أعطينا العلاقة بين AG(R) وΓ(R) .

الكلمات المفتاحية: بيان تالف المثاليات، بيان قاسم الصفر، الحلقات المختزلة، الحلقات المنتهية المحلية، الحلقات الصحيحة معيار n .

1. Introduction:

Let R be a commutative ring with identity , and let Z(R) be its set of zero divisors. We associate a simple graph Γ(R) to R with vertices $Z^*(R)=Z(R)\setminus\{(0)\}$, the set of all non-zero zero divisors of R , and for distinct $x, y \in Z^*(R)$, the vertices x and y are adjacent if and only if xy=0. Thus , Γ(R) is empty graph iff R is an integral domain.

Beck introduced the concept of zero divisor graph of a commutative ring in [4]. In the recent years zero divisor graph have been extensively studied by many authors in [1,2,3,8].

An ideal I of R is said to be annihilating ideal if there exists a non-trivial ideal J of R such that I J=(0). Let AG(R) be the set of annihilating ideals of R. The annihilating ideal graph AG(R) is a graph with vertex set $AG^*(R)=AG(R)\setminus\{(0)\}$ such that there is an edge between vertices I and J if and only if $I\neq J$ and $IJ=(0)$. The idea of annihilating ideal graph was introduced by Behboodi and Rakeei in [5] and [6].

In the present paper , we investigate the annihilating ideal graph AG(R). We establish a new of its basic properties and its relation of $\Gamma(R)$.

Recall that:

1. R is called reduced if R has no non-zero nilpotent element.
2. The distance d(u,v) between two vertices u and v of a connected graph Γ is the minimum of the lengths of the u—v paths of Γ [7].
3. The degree of the vertex a in the graph Γ is the number of edges of Γ incident with a [7].
4. The graph Γ is called a plane graph if it can be drawn in the plane with their edges crossing. A graph which is an isomorphic to a plane graph is called a planer graph[7].
5. A graph Γ is bipartite graph , if it is possible to partition the vertex set of Γ into two subsets V_1 and V_2 such that every element of edges of Γ joins a vertex of V_1 to a vertex of V_2. A complete bipartite graph with partite sets V_1 and V_2 where , $|V_1|$=m and $|V_2|$=n , is then denoted by $K_{m,n}$[7].

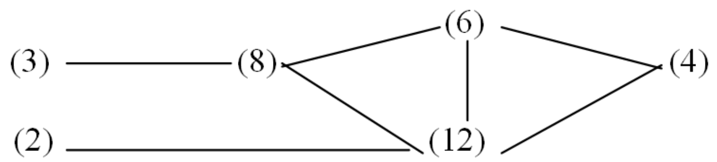
2. Annihilating ideal graph:

In this section , we consider annihilating ideal graph, we give some of its basic properties and provide some examples.

Definition2.1[5]: Let R be a ring and let I and J are distinct non-trivial ideals of R.Then , I and J are adjacent ideal vertices in AG(R) if I J=(0).

From now on , we shall use the symbol I— J to denote for two adjacent ideal vertices I and J. We start this section with the following example .

Example1: Let Z_{24} be the ring of integers modulo 24. The graph $AG(Z_{24})$ can be drawn as follows:



The following result is an easy consequence of definition of 2.1.

Lemma2.2: If I and J are non-trivial ideals of R such that $I\cap J=(0)$, then I— J is an edge of AG(R) and $I\cup J\subseteq Z(R)$.

The converse of Lemma2.2 is not true in general, as the following example shows.

Example2: Let Z_{12} be the ring of integers modulo 12.Then , (2) ——— (6) is an edge of the graph $AG(Z_{12})$, but $(2)\cap(6)\neq(0)$.

We now give a sufficient condition for the converse of Lemma2.2 to be true.

Proposition2.3: Let R be a reduced ring , and let I— J be an edge in $AG(R)$. Then , $I\cap J=(0)$.

Proof: Let $a \in I \cap J$. Then , $a \in I$ and $a \in J$, this implies that $a^2 \in I\,J = (0)$, so $a^2 = 0$. Since , R is a reduced ring , then a=0. Therefore , $I \cap J = (0)$. ■

The next result illustrates that the distance of any two nilpotent ideal vertices of AG(R) is at most 2.

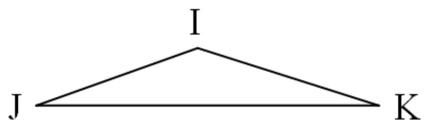
Theorem2.4: Let I and J be two ideal vertices of AG(R). If either I or J is a nilpotent , then d(I,J)≤2.
Proof: Let d(I,J)=3. Then, there is a path from I to J in AG(R) say I— K— L— J. Let I be a nilpotent ideal of R. Then , there exists an integer n>1 such that $I^n=(0)$. Consider the sequence L , LI , $LI^2,\ldots, LI^n$. Let m be the smallest integer in which $LI^m \neq (0)$. Hence , $LI^{m+1} = (0)$. Obviously , LI^m adjacent to both I and J. This contradict the fact that d(I,J)=3. Therefore , d(I,J)≤2. ■

The next result illustrates the degree of a vertex adjacent to the set of zero divisors of R.

Proposition2.5: Let R be a finite ring and let Z(R) be an ideal of R. If I— Z(R) is an edge in AG(R) , then deg(I)=|AG(R)|-1.
Proof: Suppose that I— Z(R) be an edge in AG(R) , it follows that I·Z(R)= (0). Let J be any vertex of AG(R).Then , by Lemma2.2 , J is a subset of Z(R). This implies that I·J= (0).Thus, I is adjacent to all vertices of AG(R).This means that deg(I)=|AG(R)|-1.■

Example3: Let Z_{16} be the ring of integers modulo 16. The vertices of AG(Z_{16}) are I=(8)

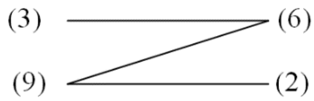
, J=(4) and K=(2)= Z(Z_{16}). Clearly deg(I)= deg(J)= |AG(Z_{16})|=3-1.



The next result considers the adjacency of two minimal ideals in the graph AG(R).

Proposition2.6: Every two distinct minimal ideals of R are adjacent in AG(R).
Proof: Let M and N be two distinct minimal ideals of R. Since , M and N contain MN , then MN=M=N or MN=(0). The first case is not true because M and N are distinct ideals. Thus , MN=(0). This means that M and N are adjacent vertices in AG(R). ■

Example4: Let Z_{18} be the ring of integers modulo 18.



Clearly , the minimal ideals of Z_{18} are (6) and (9) , which are adjacent vertices in AG(Z_{18}).

The next result considers the number of minimal ideals of R .

Theorem2.7: If $AG(R)$ is a planar graph , then R has at most four minimal ideals.
Proof: Suppose that R has five minimal ideals say M_1 , M_2 , M_3 , M_4 and M_5. By Proposition2.6 , any two of M_1 , M_2 , M_3 , M_4 and M_5 are adjacent. This means that

$AG(R)$ contains the complete graph K_5. This is contradiction that $AG(R)$ is a planar graph (See the Kuratowsky Theorem in [7]). Therefore, R has at most four minimal ideals. ∎

Example5: Let Z_{16} be the ring of integers modulo 16.

Clearly, the graph $AG(Z_{16})$ is a planar graph and the only minimal ideal of Z_{16} is (8).

3. The graphs $\Gamma(R)$ and $AG(R)$

In this section, we consider the relationship between $\Gamma(R)$ and $AG(R)$.

It is natural to ask whether $\Gamma(R)$ and $AG(R)$ are isomorphic , the answer is negative , as the following example shows.

Example6: Let Z_{12} be the ring of integer modulo 12. Then, the number of vertices of $\Gamma(Z_{12})$ is 7 , while the number of vertices of $AG(Z_{12})$ is 4. Obviously, $\Gamma(Z_{12})$ and $AG(Z_{12})$ are not isomorphic.

The next result explores the relation between the set of zero divisors of R and the vertices of AG(R).

Theorem3.1: For any ring R , $Z(R)=\cup\{I: I$ is an ideal vertex of $AG(R)\}$.

Proof: Let $0\neq x\in Z(R)$. Then, there exists $y\in Z^*(R)$ such that $xy=0$.This implies that $(x)(y)=(0)$. If $(x)=R$,then x is a unit element. This contradicts the fact that $x\in Z^*(R)$. So, $(x)\neq R$. Since (x) is adjacent to (y), then $x\in (x)\in \{I: I$ is an ideal vertex of $AG(R)\}$. Therefore, $x\in\cup\{I: I$ is an ideal vertex of $AG(R)\}$.

Conversely , suppose that $x\in\cup\{I: I$ is an ideal vertex of $AG(R)\}$ .Then, $x\in I$ for some vertex I of $AG(R)$. By Lemma2.2,$x\in Z(R)$.Hence, $Z(R)=\cup\{I: I$ is a vertex of $AG(R)\}$. ∎

Let us give the following easy result.

Proposition3.2: Let $\Gamma(R)$ and $AG(R)$ are finite graphs , then $|\Gamma(R)|\geq|AG(R)|$.

The following result demonstrates the isomorphism between $\Gamma(R)$ and AG(R) by considering $R=Z_n$.

Theorem3.3: Let $n>1$ be a non-prime integer. Then, $\Gamma(Z_n)$ contains a subgraph which isomorphic with $AG(Z_n)$.

Proof: Define the graph G by $G=\{a—b: a—b$ is an edge in $\Gamma(Z_n), a|n, b|n$ and $a\neq b\}$. Obviously, G is a subgraph of $\Gamma(Z_n)$. Now, define a function $f: G\rightarrow AG(Z_n)$by $f(a)=(a)$, with $a\in G$. Clearly f is onto. Now, for any distinct vertices $a, b\in G$, $a|n$ and $b|n$. So, $f(a)=(a)\neq(b)=f(b)$. Thus, f is one to one. Now, suppose that $a—b$ is an edge in G. Then, $ab=0$, so $(a)(b)=(0)$. This shows that $f(a)f(b)=(0)$, and hence $f(a)—f(b)$ is an edge in $AG(Z_n)$. Thus f preserves the adjacency property. This proves that $G\cong AG(Z_n)$. ∎

The following result gives a sufficient conditions for two vertices of $\Gamma(R)$ such that their annihilators are adjacent ideal vertices in $AG(R)$.

Theorem3.4: If a and b are two vertices in $\Gamma(R)$ such that $d(a,b)=3$, then $Ann(a)$ and $Ann(b)$ are adjacent ideal vertices in $AG(R)$.

Proof: Since, $a, b \in Z^*(R)$, then both Ann(a) and Ann(b) are non-zero. On the other hand , d(a,b)=3. This means that neither b∈ Ann(a) nor $a \in$ Ann(b). Then, neither Ann(a)=R nor Ann(b)=R. So, both Ann(a) and Ann(b) are nontrivial ideals. If we assume that Ann(a)Ann(b)$\neq (0)$, then there exists c∈Ann(a) and d∈Ann(b) such that cd$\neq$ 0. Clearly a(cd)=b(cd)=0. This means that $a — cd — b$ is a path in Γ(R).This contradicts the fact that d(a,b)=3.Therefore, Ann(a) and Ann(b) are adjacent ideal vertices in AG(R). ■

Example7: Let Z_{12} be the ring of integers modulo 12. Clearly , d((3),(10))=3 in AG(Z_{12}) and Ann(3)Ann(10)=(4)(6)=(0).This means that Ann(3) and Ann(10) are adjacent in AG(Z_{12}).

We end this paper by showing that,

Proposition3.5: If R is a finite local ring , then $AG(R) \neq K_{mn}$ for any integers m,n>1.
Proof: Suppose that $AG(R) = K_{mn}$for some integers m,n>1, and let A={I_1,I_2,…,I_n} and B={J_1,J_2,…,J_m} be the partition of $AG(R)$. Since, R is a local ring , then by Theorem1.2 in [9] , Z(R) is an ideal of R and there exists a vertex a of $\Gamma(R)$ such that a·Z(R)=(0). It follows that (a) ·Z(R)=(0). Hence, Z(R) is a vertex of $AG(R)$, yielding $Z(R) \in A$ or $(R) \in B$. Now, if $(R) \in A$, then J_i Z(R)=(0) for all i=1,2,…,m.By Theorem3.1, J_i J_k=(0) for i≠k. This contradicts the fact that J_i and J_k are not adjacent. If $(R) \in B$, this will lead to a contradiction. Thus, $\mathrm{AG(R)} \neq \mathrm{K_{mn}}$ for any integers m,n>1. ■

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