A Geometric Construction of Complete (k_r ,r)-arcs in PG(2,7) and the Related projective $[n,3,d]_7$ Codes

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ABSTRACT

A (k ,r)-arc is a set of k points of a projective plane PG(2,q) such that some r, but no r + 1 of them, are collinear. The (k ,r)-arc is complete if it is not contained in a (k + 1,r)-arc.

In this paper we give geometrical construction of complete (k_r ,r)-arcs in PG(2,7), r = 2,3,…, 7, and the related projective $[n,3,d]_7$ codes.

***Keywords**: Projective plane, complete arcs, codes.*

في PG(2,7) وعلاقتها بالشفرات $[n,3,d]_7$ البناء الهندسي للأقواس الاسقاطية-(k_r ,r) التامة

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الملخص

القوس-(k ,r) هو مجموعة k من النقاط في المستوي الاسقاطي PG(2,q) بحيث يوجد r من النقاط على خط ولا يوجد r+1 أو أكثر من تلك النقاط على خط, ويقال أن القوس-(k ,r) تامٌ إن لم يكن محتوياً في قوس- (k+1,r).

في هذا البحث أعطينا التركيب الهندسي للأقواس-(k_r,r) التامة في المستوي الاسقاطي PG(2,7), r=2,3,…,7 , وعلاقته بالشفرات $[n,3,d]_7$ الأسقاطية.

الكلمات المفتاحية : المستوي الاسقاطي، القوس التام ،الشفرات.

1. Introduction

Let PG(2,q) be the projective plane over Galois field GF(q). The points of PG(2,q) are the non-zero vectors of the vector space V(3,q) with the rule that X(x_1,x_2,x_3) and Y(λx_1,λx_2,λx_3) are the same point, where $\lambda \in$ GF(q)\{0}.

Similarly, x[x_1,x_2,x_3] and y[(λx_1,λx_2,λx_3] are the same line, where $\lambda \in$ GF(q)\{0}. The point X(x_1,x_2,x_3) is on the line Y[y_1,y_2,y_3] if and only if $x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$. The number of points and the number of lines in PG(2,q) is $q^2 + q + 1$ .There are q + 1 points on every line and q+1 lines through every point [1,2,3,4,5,6] .

Definition 1.1:[1]

A (k ,r)-arc K is in PG(2,q) is a set of k points such that some line of the plane meets K in n points but such that no line meets K in more than r points, where $r \geq 2$.

Definition 1.2:[1]

A (k ,r)-arc is complete if it is not contained in a (k+1,r)-arc. The maximum number of points that a (k,2)-arc can have is m(2,q) and this arc is an oval.

Theorem 1.3:[1]

In PG(2,q), $m(2,q)=\begin{cases} q+1 & \text{for q odd} \\ q+2 & \text{for q even} \end{cases}$

Definition 1.4:[3]

A line ℓ in PG(2,q) is an i-secant of a (k ,r)-arc K if $|\ell \cap K| = i$.

Definition 1.5:[3]

A variety V(F) of PG(2,q) is a subset of PG(2,q) such that

V(F) = {P(X) ∈ PG(2,q) | F(X) = 0}. Where F(X) is a homogenous polynomial F in three variables x_1,x_2,x_3 over F_q , P(X) is the point of PG(2,q) represented by X=(x_1,x_2,x_3).

Definition 1.6:[3]

Let Q(2,q) be the set of quadrics in PG(2,q), that is the varieties V(F), where:

$$F = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3 \qquad \ldots(1)$$

If V(F) is non singular, then the quadric is a conic, that is, if

$$A = \begin{bmatrix} a_{11} & \frac{a_{12}}{2} & \frac{a_{13}}{2} \\ \frac{a_{12}}{2} & a_{22} & \frac{a_{23}}{2} \\ \frac{a_{13}}{2} & \frac{a_{23}}{2} & a_{33} \end{bmatrix}$$

is non singular, then the quadric (1) is a conic.

Theorem 1.7:[3]

Every conic in PG(2,q) is a (q + 1) -arc.

Theorem 1.8:[3]

In PG(2,q), with q odd, every oval is a conic.

Definition 1.9:[3]

A point N which is not on a (k ,r)-arc has index if there exactly i(n-secants) of the arc through N, the number of the points N of index i is denoted by N_i.

Remark 1.10:[3]

The (k ,r)-arc is complete if and only if $N_0 = 0$. Thus, the arc is complete if and only if every point of PG(2,q)is not on the arc lies an some n-secant of the arc.

Definition 1.11:[4,5]

An (b ,t)-blocking set B in PG(2,q) is a set of b points such that every line of PG(2,q) intersects B in at least t points, and there is a line intersecting B in exactly t points.

If B contains a line, it is called trivial, thus B is a subset of PG(2,q) which meets every line but contains no line completely; that is $t \le |B \cap \ell| \le q$ for every line ℓ in PG(2,q). So, B is a blocking set if and only if PG(2,q)\B is.**[1]** We may note that a blocking set is merely a (k,r)-arc with $r \le q$ and no 0-secants. Note that a (k ,r)-arc is the complement of (q^2 + q + 1 – k ,q + 1 – r)-blocking set in PG(2,q) and conversely. A blocking set B is minimal if B\{p} is not blocking set for every p ∈ B.

Definition 1.12:[1]

Let B be a set contains a line ℓ minus a point p plus a set of q points, one on each of the q lines through P other than ℓ but not all collinear; b = 2q, then B is minimal blocking set. Blocking sets of this kind are called Redei-type studied by [Bruen, A.A. and Thas, J.A.(1977)] and in [Blokhuis, A.A. and Brouwer, E.and S.Z. "onyi, T.(1995)].

Definition 1.13:[5]

Let V(n,q) denote the vector space of all ordered n-tuples over GF(q). A linear code C over GF(q) of length n and dimension k is a k-dimensional subspace of V(n,q). The vectors of C are called codewords. The Hamming distance between two codewords is defined to be the number of coordinate places in which they differ. The minimum distance of a code is the smallest distances between distinct codewords. Such a code is called an $[n,k,d]_q$ code if its minimum hamming distance is d.

There exists a relationship between (n ,r)-arcs in PG(2,q) and $[n,3,d]_q$ codes, given by the next theorem.

Theorem 1.14:[6]

There exists a projective $[n,3,d]_q$ code if and only if there exists an (n ,n – d)-arc in PG(2,q).

A projective plane π = PG(2,7) over GF(7) consists of 57 points, 57 lines, each line contains 8 points and through every point there are 8 lines.

Let P_i and L_i be the points and lines of PG(2,7), respectively. Let i stands for the point P_i, i = 1,2,…, 57. The points and the lines of PG(2,7) are given in omit, table (1).

Definition 1.15:[3]

The points in PG(2,p) have a unique forms which are (1,0,0), (x,1,0), (x,y,1)and(1,1,1)

for all x, y in GF(p).

There exists one point of the form (1,0,0),
There exists p points of the form (x,1,0),
There exists p^2 points of the form (x,y,1),

When(x ,y=0) then the points in PG(2,p) are called reference points .

There exists one point of the form (1,1,1) is called unit point.

2- The Constructions

Let A ={1,2,9,17} be the set of reference and unit points, where 1= (1,0,0) , 2= (0,1,0), 9= (0,0,1), 17= (1,1,1).[see table(1)]

A is a (4,2)-arc since no three points of A are collinear. There are twenty points of index zero for A, which are: 26,27,28,29,32,34,35,36,39,40,42,43,46,47,48,50,53,54,55 and 56 . Hence, A is incomplete (4,2)-arc.

2.1 The Conics in PG(2,7) through the Reference and Unit Points

The general equation of the conic is:

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_1x_2 + a_5x_1x_3 + a_6x_2x_3 = 0 \qquad \ldots(1)$$

By substituting the points of A in (1), we get;

$a_1 = a_2 = a_3 = 0$ and

$a_4 + a_5 + a_6 = 0$ so (1) becomes:

$$a_4x_1x_2 + a_5x_1x_3 + a_6x_2x_3 = 0 \qquad \ldots(2)$$

If $a_4 = 0$, then the conic is degenerated, therefore $a_4 \neq 0$, similarly, $a_5 \neq 0$ and $a_6 \neq 0$. Dividing equation (2) by a_4, we get:

$x_1x_2 + \alpha x_1x_3 + \beta x_2x_3 = 0$ where $\alpha = \frac{a_5}{a_4}$, $\beta = \frac{a_6}{a_4}$, then $\beta = -(1+\alpha)$

since $1 + \alpha + \beta = 0 \pmod 7$.

$\alpha \neq 0$ and $\alpha \neq 6$, for if $\alpha = 0$ or $\alpha = 6$ we get a degenerated conic, thus, $\alpha = 1,2,3,4,5$ and (2) can be written as:

$$x_1x_2 + \alpha x_1x_3 - (1+\alpha)x_2x_3 = 0 \qquad \ldots(3)$$

2.2 The Equations and the Points of the Conics in PG(2,7) through the Unit and Reference Points

For any value of α, there is a unique conic contains 8 points as the following

1. If $\alpha = 1$, then the equation of the conic C_1 is $x_1x_2 + x_1x_3 + 5x_2x_3 = 0$
The points of C_1 are {1,2,9,17,29,35,40 and 48}.

2. If $\alpha = 2$, then the equation of the conic C_2 is $x_1x_2 + 2x_1x_3 + 4x_2x_3 = 0$
The points of C_2 are{1,2,9,17,28,36,39 and 55}.

3. If $\alpha = 3$, then the equation of the conic C_3 is $x_1x_2 + 3x_1x_3 + 3x_2x_3 = 0$
The points of C_3 are {1,2,9,17,26,32,50 and 56}.

4. If $\alpha = 4$, then the equation of the conic C_4 is $x_1x_2 + 4x_1x_3 + 2x_2x_3 = 0$
The points of C_4 are {1,2,9,17,27,43,46 and 54}.

5. If $\alpha = 5$, then the equation of the conic C_5 is $x_1x_2 + 5x_1x_3 + x_2x_3 = 0$
The points of C_5 are {1,2,9,17,34,42,47 and 53}.

Thus ,we found five maximum complete (k,2)-arcs C_1, C_2, C_3, C_4 and C_5.

2.3 The Construction of Complete (k_r, r)-arcs in PG(2,7)

The complete (k,n)-arcs in PG(2,7) can be constructed by eliminating the conics given above from the projective plane PG(2,7) as follows:

2.3.1 The Construction of Complete (k_7,7)-arc and the related projective $[43,3,36]_7$ codes

We take one conic, say C_1, and let $K = \pi - C_1$, C_1 = {1,2,9,17,29,35,40,48}= {3,4,5,6,7,8,10,11,12,13,14,15,16,18,19,20,21,22,23,24,25,26,27,28,30,31,32,33,34,36, 37,38,39,41,42,43,44,45,46,47,49,50,51,52,53,54,55,56 and 57}

The construction must satisfy the following:

(1) K intersects any line of π in at most 7 points.
(2) Every point not in K is on at least one 7-secant of K.

The points: 8,16,20,21,30,44,49,51 and 57 are eliminated from K to satisfy (1). The points of index zero for K 1,17,29 are added to K to satisfy (2), then

K_7 = K $\cup$ {1,17,29}\{8,16,20,21,30,44,49,51 and 57} Thus K_7 = {1,3,4,5,6,7,10,11,12,13,14,15,17,18,19,22,23,24,25,26,27,28,29,31,32,33,34,36, 37,38, 39,41,42,43,45,46,47,50,52,53,54,55 and 56} is a complete (43,7)-arc as shown in table (2).

Let $\beta_1 = \pi - K_7$ = {2,8,9,16,20,21,30,35,40,44,48,49,51 and 57} is (14,1)-blocking set as shown in table (2). β_1 is of Redei-type contains the line ℓ_1 = {2,9,16,23,37,44,51,30}\{37} and one point on each line through the point 37 which are non-collinear points: 40,35,20,57,48,8 and 49.By theorem (1.14), there exists a projective [43,3,36]$_7$ code which is equivalent to the complete (43,7)-arc K_7.

2.3.2 The Construction of Complete (k_6,6)-arc and the related projective [35,3,29]$_7$ codes

We take two conics, say C_1 andC_2,C_1 = {1,2,9,17,29,35,40 and 48}, C_2 = {1,2,9,17,28,36,39 and 55},

let K=π–$C_1 \cup C_2$,= {3,4,5,6,7,8,10,11,12,13,14,15,16,18,19,20,21,22,23,24,25,26,27,30, 31,32,33,34,37,38, 41,42,43,44,45,46,47,49,50,51,52,53,54,56,57}

The construction must satisfy the following:

(1) K intersects any line of π in at most 6 points.
(2) Every point not in K is on at least one 6-secant of K.

The points: 4,5,8,10,16,19,23,30,37,42,44 and 56 are eliminated from K to satisfy (1). The points 35, 55 are added to K to satisfy (2), then

K_6 = K $\cup$ {35,55}\{4,5,8,10,16,19,23,30,37,42,44 and 56}

={3,6,7,11,12,13,14,15,18,20,21,22,24,25,26,27,31,32,33,34,35,38,41,43,45,46, 47,49,50, 51,52,53,54,55 and 57}

K_6 is a complete (35,6)-arc, then shows table (3)

β_2 = {1,2,4,5,8,9,10,16,17,19,23,28,29,30,36,37,39,40,42,44,48 and 56} is a (22,2)-blocking set as shown in table (3). By theorem (1.14), there exists a projective [35,3,29]$_7$ code which is equivalent to the complete (35,6)-arc K_6.

2.3.3 The Construction of Complete (k_5,5)-arc and the related projective [27,3,22]$_7$ codes

We take the union of three conics, say C_1, C_2 and C_3,

$C_1 \cup C_2 \cup C_3$ = {1,2,9,17,26,28,29,32,35,36,39,40,48,50,55 and 56}.

Let K = $\pi - C_1 \cup C_2 \cup C_3$ = {3,4,5,6,7,8,10,11,12,13,14,15,16,18,19,20,21,22,23, 24,25,27,30,31,33,34,37,38,41,42, 43,44,45,46,47,49,51,52,53,54 and 57}

The construction must satisfy the following:

(1) K intersects every line of π in at most 5 points.
(2) Every point not in K is on at least one 5-secant of K.

The points: 3,5,6,7,8,11,15,16,18,20,23,24,33,34,37,42,51,52 and 54 are eliminated from K in order to satisfy (1). The points 17,32,35,56 are added to K to satisfy (2), then

K_5 = K $\cup$ {17,32,35,56}\{3,5,6,7,8,11,15,16,18,20,23,24,33,34,37,42,51,52 and 54}

= {4,10,12,13,14,17,19,21,22,25,27,30,31,32,35,38,41,43,44,45,46,47,49,53, 56 and 57}

is a complete (26,4)-arc, then shows table (4)

β_3={1,2,3,5,6,7,8,9,11,15,16,18,20,23,24,26,28,29,33,34,36,37,39,40,42,48,50,51,52,54, 55} is a (31,3)-blocking set as shown in table (4). By theorem (1.14), there exists a projective $[27,3,22]_7$ code which is equivalent to the complete (26,4)-arc K_5.

2.3.4 The Construction of Complete (k_4,4)-arc and the related projective $[17,3,13]_7$ codes

We take the union of four conics, say C_1, C_2, C_3 and C_4,

$C_1 \cup C_2 \cup C_3 \cup C_4$ = {1,2,9,17,26,27,28,29,32,35,36,39,40,43,46,48,50,54,55 and 56}.

Let $K = \pi - C_1 \cup C_2 \cup C_3 \cup C_4$

= {3,4,5,6,7,8,10,11,12,13,14,15,16,18,19,20,21,22,23,24,25, 30,31,33,34,37,38

,41,42,44,45, 47,49,51,52,53 and 57}

The construction must satisfy the following:

(1) K intersects every line of π in at most 4 points.
(2) Every point not in K is on at least one 4-secant of K.

The points: 3,5,6,7,11,15,16,18,21,22,25,30,37,38,41,42,44,49,52 and 53 are eliminated from K in order to satisfy (1), and there are no points of index zero, then

K_4 = K\{3,5,6,7,11,15,16,18,21,22,25,30,37,38,41,42,44,49,52 and 53}

= {4,8,10,12,13,14,19,20,23,24,31,33,34,45,47,51 and 57}

is a complete (17,4)-arc, then

β_4 = {1,2,3,5,6,7,9,11,15,16,17,18,21,22,25,26,27,28,29,30,32,35,36,37,38,39,40,41,42,43,44 , 46,48,49,50,52,53,54,55 and 56} is a (40,4)-blocking set which is trivial since β_4 contains some lines of π as shown in table (5). By theorem (1.14), there exists a projective $[17,3,13]_7$ code which is equivalent to the complete (17,4)-arc K_4.

2.3.5 The Construction of Complete (k_3,3)-arc and the related projective $[12,3,9]_7$ codes

We take the union of five conics C_1, C_2 ,C_3, C_4 and C_5, then

$C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$= {1,2,9,17,26,27,28,29,32,34,35,36,39,40,42,43,46,47,48,50,53,54,55 and 56}.

Let $K = \pi - C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$={3,4,5,6,7,8,10,11,12,13,14,15,16,18,19,20, 21,22,23,24,25,30,31,33,37,38,41,44,45,49, 51,52 and 57}

The construction must satisfy the following:

(1) K intersects every line in π in at most 3 points.
(2) Every point not in K is on at least one 3-secant of K.

Thepoints:3,4,5,7,8,10,11,12,13,14,16,19,20,21,22,23,25,30,31,33,37,38 and 41,44are eliminated from K to satisfy (1), and the points 9, 26, 47 are added to K to satisfy (2), then

$K_3 = K \cup \{9, 26, 47\} \backslash \{3,4,5,7,8,10,11,12,13,14,16,19,20,21,22,23,25,30,31,33,37, 38,41,44\} = \{6,9,15,18,24,26,45,47,49,51,52$ and $57\}$is a complete (12,3)-arc as shown in table (6), then

$\beta_5 = \pi - K_3 = \{1,2,3,4,5,7,8,10,11,12,13,14,16,17,19,20,21,22,23,25,27,28,29,30,31,32,33, 34,35,36,37, 38,39,40,41,42,43,44,46,48,50,53,54,55$ and $56\}$ is a (45,5)-blocking set which is a trivial since β_5 contains some lines of π as shown in table (6). By theorem (1.14), there exists a projective $[12,3,9]_7$ code which is equivalent to the complete (12,3)-arc K_3.

2.3.6 The Construction of Complete (k_2,2)-arc and the related projective $[6,3,4]_7$ codes

The construction must satisfy the following:

(1) The complete arc intersects every line in π in at most 2 points.
(2) Every point not in the arc is on at least one 2-secant of the arc.

To construct a complete K_2 arc, we eliminate the points 15,18,24,26,47,51 and 57 from K_3 to satisfy (1), and add the point 4 to satisfy (2), then

$K_2 = K_3 \cup \{4\} \backslash \{15,18,24,26,47,51$ and $57\} = \{4,6,9,45,49$ and $52\}$

is a complete (6,2)-arc as shown in table (7), then

$\beta_6 = \pi \setminus K_2$

$= \{1,2,3,5,7,8,10,\ldots,44,46,47,48,50,51,53,\ldots,57\}$ is a (51,6)-blocking set which is a trivial since β_6 contains some lines of π. By theorem (1.14), there exists a projective $[6,3,4]_7$ code which is equivalent to the complete (6,2)-arc K_2.

3. Conclusions:

1- We obtain five conics in PG(2,7).

2- We construct complete (k_7,7)-arc by eliminating one conic, a (k_6,6)-arc by eliminating two conics, a (k_5,5)-arc by eliminating three conics, a (k_4,4)-arc by eliminating four conics, a (k_3,3)-arc by eliminating five conics. Note that in each step we eliminate some points from each set and adding some points to the set such that the set is a complete arc.

3- We construct projective [n,3,d]7 codes equivalent to each of these arcs..

Table (1)Points and Lines of PG(2,7)

i	***P_i***			***ℓ_i***							
1	1	0	0	2	9	16	23	30	37	44	51
2	0	1	0	1	9	10	11	12	13	14	15
3	1	1	0	8	9	22	28	34	40	46	52
4	2	1	0	5	9	19	29	32	42	45	55
5	3	1	0	4	9	18	27	36	38	47	56
6	4	1	0	7	9	21	26	31	43	48	53
7	5	1	0	6	9	20	24	35	39	50	54

8	6	1	0	3	9	17	25	33	41	49	57
9	0	0	1	1	2	3	4	5	6	7	8
10	1	0	1	2	15	22	29	36	43	50	57
11	2	0	1	2	12	19	26	33	40	47	54
12	3	0	1	2	11	18	25	32	39	46	53
13	4	0	1	2	14	21	28	35	42	49	56
14	5	0	1	2	13	20	27	34	41	48	55
15	6	0	1	2	10	17	24	31	38	45	52
16	0	1	1	1	51	52	53	54	55	56	57
17	1	1	1	8	15	21	27	33	39	45	51
18	2	1	1	5	12	22	25	35	38	48	51
19	3	1	1	4	11	20	29	31	40	49	51
20	4	1	1	7	14	19	24	36	41	46	51
21	5	1	1	6	13	17	28	32	43	47	51
22	6	1	1	3	10	18	26	34	42	50	51
23	0	2	1	1	30	31	32	33	34	35	36
24	1	2	1	7	15	20	25	30	42	47	52
25	2	2	1	8	12	18	24	30	43	49	55
26	3	2	1	6	11	22	26	30	41	45	56
27	4	2	1	5	14	17	27	30	40	50	53
28	5	2	1	3	13	21	29	30	38	46	54
29	6	2	1	4	10	19	28	30	39	48	57
30	0	3	1	1	23	24	25	26	27	28	29
31	1	3	1	6	15	19	23	34	38	49	53
32	2	3	1	4	12	21	23	32	41	50	52
33	3	3	1	8	11	17	23	36	42	48	54
34	4	3	1	3	14	22	23	31	39	47	55
35	5	3	1	7	13	18	23	35	40	45	57
36	6	3	1	5	10	20	23	33	43	46	56
37	0	4	1	1	44	45	46	47	48	49	50
38	1	4	1	5	15	18	28	31	41	44	54
39	2	4	1	7	12	17	29	34	39	44	56
40	3	4	1	3	11	19	27	35	43	44	52
41	4	4	1	8	14	20	26	32	38	44	57
42	5	4	1	4	13	22	24	33	42	44	52
43	6	4	1	6	10	21	25	36	40	44	55
44	0	5	1	1	37	38	39	40	41	42	43
45	1	5	1	4	15	17	26	35	37	46	55
46	2	5	1	3	12	20	28	36	37	45	53
47	3	5	1	5	11	21	24	34	37	47	57
48	4	5	1	6	14	18	29	33	37	48	52

49	5	5	1	8	13	19	25	31	37	50	56
50	6	5	1	7	10	22	27	32	37	49	54
51	0	6	1	1	16	17	18	19	20	21	22
52	1	6	1	3	15	16	24	32	40	48	56
53	2	6	1	6	12	16	27	31	42	46	57
54	3	6	1	7	11	16	28	33	38	50	55
55	4	6	1	4	14	16	25	34	43	45	54
56	5	6	1	5	13	16	26	36	39	49	52
57	6	6	1	8	10	16	29	35	41	47	53

Table (2)

i	**$K_7 \cap \ell_i$**	**$\beta_1 \cap \ell_i$**
1	{37}	{2,9,16,23,30,44,51}
2	{1,10,11,12,13,14,15}	{9}
3	{22,28,34,46,52}	{8,9,40}
4	{5,19,29,32,42,45,55}	{9}
5	{4,18,27,36,38,47,56}	{9}
6	{7,21,26,31,43,53}	{9,48}
7	{6,24,39,50,54}	{9,20,35}
8	{3,17,25,33,41}	{9,49,57}
9	{1,3,4,5,6,7}	{2,8}
10	{15,22,29,36,43,50}	{2,57}
11	{12,19,26,33,47,54}	{2,40}
12	{11,18,25,32,39,46,53}	{2}
13	{14,21,28,42,56}	{2,35,49}
14	{13,27,34,41,55}	{2,20,48}
15	{10,17,24,31,38,45,52}	{2}
16	{1,52,53,54,55,56}	{51,57}
17	{15,21,27,33,39,45}	{8,51}
18	{5,12,22,25,38}	{35,48,51}
19	{4,11,29,31}	{20,40,49,51}
20	{7,14,19,24,36,41,46}	{51}
21	{6,13,17,28,32,43,47}	{51}
22	{3,10,18,26,34,42,50}	{51}
23	{1,31,32,33,34,36}	{30,35}
24	{7,15,25,42,47,52}	{20,30}
25	{12,18,24,43,55}	{8,30,49}
26	{6,1,22,26,41,45,56}	{30}
27	{5,14,17,27,50,53}	{30,40}
28	{3,13,21,29,38,46,54}	{30}
29	{4,10,19,28,39}	{30,48,57}
30	{1,24,25,26,27,28,29}	{23}
31	{6,15,19,34,38,53}	{23,49}
32	{4,12,21,32,41,50,52}	{23}
33	{11,17,36,42,54}	{8,23,48}

34	{3,14,22,31,39,47,55}	{23}
35	{7,13,18,45}	{23,35,40,57}
36	{5,10,33,43,46,56}	{20,23}
37	{1,45,46,47,50}	{44,48,49}
38	{5,15,18,28,31,41,54}	{44}
39	{7,12,17,29,34,39,56}	{44}
40	{3,11,19,27,43,52}	{35,44}
41	{14,26,32,38}	{8,20,44,57}
42	{4,13,22,24,33,42,53}	{44}
43	{6,10,21,25,36,55}	{40,44}
44	{1,37,38,39,41,42,43}	{40}
45	{4,15,17,26,37,46,55}	{35}
46	{3,12,28,36,37,45,53}	{20}
47	{5,11,21,24,34,37,47}	{57}
48	{6,14,18,29,33,37,52}	{48}
49	{13,19,25,31,37,50,56}	{8}
50	{7,10,22,27,32,37,54}	{49}
51	{1,17,18,19,21,22}	{16,20}
52	{3,15,24,32,56}	{16,40,48}
53	{6,12,27,31,42,46}	{16,57}
54	{7,11,28,33,38,50,55}	{16}
55	{4,14,25,34,43,45,54}	{16}
56	{5,13,26,36,39,52}	{16,49}
57	{10,29,41,47,53}	{8,16,35}

Table (3)

i	$K_6 \cap \ell_i$	$\beta_2 \cap \ell_i$
1	{51}	{2,9,16,23,30,37,44}
2	{11,12,13,14,15}	{1,9,10}
3	{22,34,46,52}	{8,9,28,40}
4	{32,45,55}	{5,9,19,29,42}
5	{18,27,38,47}	{4,9,36,56}
6	{7,21,26,31,43,53}	{9,48}
7	{6,20,24,35,50,54}	{9,39}
8	{3,25,33,41,49,57}	{9,17}
9	{3,6,7}	{1,2,4,5,8}
10	{15,22,43,50,57}	{2,29,36}
11	{12,26,33,47,54}	{2,19,40}
12	{11,18,25,32,46,53}	{2,39}
13	{21,35,49}	{2,14,28,42,56}
14	{13,20,27,34,41,55}	{2,48}
15	{24,31,38,45,52}	{2,10,17}
16	{51,52,53,54,55,57}	{1,56}
17	{15,21,27,33,45,51}	{8,39}
18	{12,22,25,35,38,51}	{5,48}
19	{11,20,31,49,51}	{4,29,40}
20	{7,14,24,41,46,51}	{19,36}

21	{6,13,32,43,47,51}	{17,28}
22	{3,18,26,34,50,51}	{10,24}
23	{31,32,33,34,35}	{1,30,36}
24	{7,15,20,25,47,52}	{30,42}
25	12,18,24,43,49,55}	{8,30}
26	{6,11,22,26,41,45}	{30,56}
27	{5,14,27,50,53}	{17,30,40}
28	{3,13,21,38,46,54}	{29,30}
29	{57}	{4,10,19,28,30,39,48}
30	{24,25,26,27}	{1,23,28,29}
31	{6,15,34,38,49,53}	{19,23}
32	{12,21,32,41,50,52}	{4,23}
33	{11,54}	{8,17,23,36,42,48}
34	{3,14,22,31,47,55}	{23,39}
35	{7,13,18,35,45,57}	{23,40}
36	{20,33,43,46}	{5,10,23,56}
37	{45,46,47,49,50}	{1,44,48}
38	{15,18,31,41,54}	{5,28,44}
39	{7,21,34}	{17,29,39,44,56}
40	{3,11,27,35,43,52}	{19,44}
41	{14,20,26,32,38,57}	{8,44}
42	{13,22,24,33,53}	{4,42,44}
43	{6,21,25,55}	{10,36,40,44}
44	{38,41,43}	{1,37,39,40,42}
45	{15,26,35,46,55}	{4,17,37}
46	{3,12,20,45,53}	{28,36,37}
47	{11,21,24,34,47,57}	{5,37}
48	{6,14,18,33,52}	{29,37,48}
49	{13,25,31,50}	{8,19,37,56}
50	{7,22,27,32,49,54}	{10,37}
51	{18,20,21,22}	{1,16,17,19}
52	{3,15,24,32}	{16,40,48,56}
53	{6,12,27,31,46,57}	{16,42}
54	{7,11,3338,50,55}	{16,28}
55	{14,25,34,43,45,54}	{4,16}
56	{13,26,49,52}	{5,16,36,39}
57	{35,41,47,53}	{8,10,16,29}

Table (4)

i	$K_5 \cap \ell_i$	$\beta_3 \cap \ell_i$
1	{30,44}	{2,9,16,23,37,51}
2	{10,12,13,14}	{1,9,11,15}
3	{22,46}	{8,9,28,34,40,52}
4	{19,32,45,55}	{5,9,29,42}
5	{4,27,38,47,56}	{9,18,36}
6	{21,31,43,53}	{7,9,26,48}

7	{35}	{6,9,20,24,39,50,54}
8	{17,25,41,49,57}	{3,9,33}
9	{4}	{1,2,3,5,6,7,8}
10	{22,43,57}	{2,15,29,36,50}
11	{12,19,47}	{2,26,33,40,54}
12	{25,32,46,53}	{2,11,18,39}
13	{14,21,35,49,56}	{2,28,42}
14	{13,27,41,55}	{2,20,34,48}
15	{10,17,31,38,45}	{2,24,52}
16	{53,55,56,57}	{1,51,52,54}
17	{21,27,45}	{8,15,33,39,51}
18	{12,22,25,35,38}	{5,48,51}
19	{4,31,49}	{11,20,29,40,51}
20	{14,19,41,46}	{7,24,36,51}
21	{13,17,32,43,47}	{6,28,51}
22	{10}	{3,18,26,34,42,50,51}
23	{30,31,32,35}	{1,33,34,36}
24	{25,30,47}	{7,15,20,42,52}
25	{12,30,43,49,55}	{8,18,24}
26	{22,30,41,45,56}	{6,11,26}
27	{14,17,27,30,53}	{5,40,50}
28	{13,21,30,38,46}	{3,29,54}
29	{4,10,19,30,57}	{28,39,48}
30	{25,27}	{1,23,24,26,28,29}
31	{19,38,49,53}	{6,15,23,34}
32	{14,12,21,32,41}	{23,50,52}
33	{17}	{8,11,23,36,42,48,54}
34	{14,22,31,47,55}	{3,23,39}
35	{13,35,45,57}	{7,18,23,40}
36	{10,43,46,56}	{5,20,23,33}
37	{44,45,46,47,49}	{1,48,50}
38	{31,41,44}	{5,15,18,28,54}
39	{12,17,44,56}	{7,29,34,39}
40	{19,27,35,43,44}	{3,11,52}
41	{14,32,38,44,57}	{8,20,26}
42	{4,13,22,44,53}	{24,33,42}
43	10,21,25,44,55}	{6,36,40}
44	{38,41,43}	{1,37,39,40,42}
45	{4,17,35,46,55}	{15,26,37}
46	{12,45,53}	{3,20,28,36,37}
47	{21,47,57}	{5,11,24,34,37}
48	{14}	{6,18,29,33,37,48,52}
49	{13,19,25,31,56}	{8,37,50}
50	{10,22,27,32,49}	{7,37,54}
51	{17,19,21,22}	{1,16,18,20}
52	{32,56}	{3,15,16,24,40,48}
53	{12,27,31,46,57}	{6,16,42}

54	{38,55}	{7,11,16,28,33,50}
55	{4,14,25,43,45}	{16,34,54}
56	{13,49}	{5,16,26,36,39,52}
57	{10,35,41,47,53}	{8,16,29}

Table (5)

i	$K_4 \cap \ell_i$	$\beta_4 \cap \ell_i$
1	{51}	{2,9,16,23,30,37,44}
2	{10,12,13,14}	{1,9,11,15}
3	{8,34}	{9,22,28,40,46,52}
4	{19,45}	{5,9,29,32,42,55}
5	{4,47}	{9,18,27,36,38,56}
6	{31}	{7,9,21,26,43,48,53}
7	{20,24}	{6,9,35,39,50,54}
8	{33,57}	{3,9,17,25,41,49}
9	{4,8}	{1,2,3,5,6,7}
10	{57}	{2,15,22,29,36,43,50}
11	{12,19,33,47}	{2,26,40,54}
12	ϕ	{2,11,18,25,32,39,46,53}
13	{14}	{2,21,28,35,42,49,56}
14	{13,20,34}	{2,27,41,48,55}
15	{10,24,31,45}	{2,17,38,52}
16	{51,57}	{1,52,53,54,55,56}
17	{8,33,45,51}	{15,21,27,39}
18	{12,51}	{5,22,25,35,38,48}
19	{4,20,31,51}	{11,29,40,49}
20	{14,19,24,51}	{7,36,41,46}
21	{13,47,51}	{6,17,28,32,43}
22	{10,34,51}	{3,18,26,42,50}
23	{31,33,34}	{1,30,32,35,36}
24	{20,47}	{7,15,25,30,42,52}
25	{8,12,24}	{18,30,43,49,55}
26	{45}	{6,11,22,26,30,41,56}
27	{14}	{5,17,27,30,40,50,53}
28	{13}	{3,21,29,30,38,46,54}
29	{4,10,19,57}	{28,30,39,48}
30	{23,24}	{1,25,26,27,28,29}
31	{19,23,34}	{6,15,38,49,53}
32	{4,12,23}	{21,32,41,50,52}
33	{8,23}	{11,17,36,42,48,54}
34	{14,23,31,47}	{3,22,39,55}
35	{13,23,45,57}	{7,18,35,40}
36	{10,20,23,33}	{5,43,46,56}
37	{45,47}	{1,44,46,48,49,50}
38	{31}	{5,15,18,28,41,44,54}
39	{12,34}	{7,17,29,39,44,56}
40	{19}	{3,11,27,35,43,44,52}

41	{8,14,20,57}	{26,32,38,44}
42	{4,13,24,33}	{22,42,44,53}
43	{10}	{6,21,25,36,40,44,55}
44	ϕ	{1,37,38,39,40,41,42,43}
45	{4}	{15,17,26,35,37,46,55}
46	{12,20,45}	{3,28,36,37,53}
47	{24,34,47,57}	{5,11,21,37}
48	{14,33}	{6,18,29,37,48,52}
49	{8,13,19,31}	{25,37,50,56}
50	{10}	{7,22,27,32,37,49,54}
51	{19,20}	{1,16,17,18,21,22}
52	{24}	{3,15,16,32,40,48,56}
53	{12,31,57}	{6,16,27,42,46}
54	{33}	{7,11,16,28,38,50,55}
55	{4,14,34,45}	{16,25,43,54}
56	{13}	{5,16,26,36,39,49,52}
57	{8,10,47}	{16,29,35,41,53}

Table (6)

i	$K_3 \cap \ell_i$	$\beta_5 \cap \ell_i$
1	{9,51}	{2,16,23,30,37,44}
2	{9,15}	{1,10,11,12,13,14}
3	{9,25}	{8,22,28,34,40,46}
4	{9,45}	{5,19,29,32,40,46}
5	{9,18,47}	{4,27,36,38,56}
6	{9,26}	{7,21,31,43,48,53}
7	{6,9,24}	{20,35,39,50,54}
8	{9,49,57}	{3,17,25,33,41}
9	{6}	{1,2,3,4,5,7,8}
10	{15,57}	{2,22,29,36,43,50}
11	{26,47}	{2,12,19,33,40,54}
12	{18}	{2,11,25,32,39,46,53}
13	{49}	{2,14,21,28,35,42,56}
14	ϕ	[2,13,20,27,34,41,48,55}
15	{24,45,52}	{2,10,17,31,38}
16	{51,52,57}	{1,53,54,55,56}
17	{15,45,51}	{8,21,27,33,39}
18	{51}	{5,12,22,25,35,38,48}
19	{49,51}	{4,11,20,29,31,40}
20	{24,51}	{7,14,19,36,41,46}
21	{6,47,51}	{13,17,28,32,43}
22	{18,26,51}	{3,10,34,42,50}
23	ϕ	{1,30,31,32,33,34,35,36}
24	{15,47,52}	{7,20,25,30,42}
25	{18,24,49}	{8,12,30,43,55}
26	{6,26,45}	{11,22,30,41,56}
27	ϕ	{5,14,17,27,30,40,50,53}

28	ϕ	{3,13,21,29,30,38,46,54}
29	{57}	{4,10,19,28,30,39,48}
30	{24,26}	{1,23,25,27,28,29}
31	{6,15,49}	{19,23,34,38,53}
32	{52}	{4,12,21,23,32,41,50}
33	ϕ	{8,11,17,23,37,42,48,54}
34	{47}	{3,14,22,23,31,39,55}
35	{18,45,57}	{7,13,23,35,40}
36	ϕ	{5,10,20,23,33,43,46,56}
37	{45,47,49}	{1,44,46,48,50}
38	{15,18}	{5,28,31,41,44,54}
39	ϕ	[7,12,17,29,34,39,44,56}
40	{52}	{3,11,19,27,35,43,44}
41	{26,57}	{8,14,20,32,38,44}
42	{24}	{4,13,22,33,42,44,53}
43	{6}	{10,21,25,36,40,44,55}
44	ϕ	{1,37,38,39,40,41,42,43}
45	{15,26}	{4,17,35,37,46,55}
46	{45}	{3,12,20,28,36,37,53}
47	{24,47,57}	{5,11,21,34,37}
48	{6,18,52}	{14,29,33,37,48}
49	ϕ	{8,13,19,25,31,37,50,56}
50	{49}	{7,10,22,27,32,37,54}
51	{18}	{1,16,17,19,20,21,22}
52	{15,24}	{3,16,32,40,48,56}
53	{6,57}	{12,16,27,31,42,46}
54	ϕ	{7,11,16,28,33,38,50,55}
55	{45}	{4,14,16,25,34,43,54}
56	{26,49,52}	{5,13,16,36,39}
57	{47}	{8,10,16,29,35,41,53}

Table (7)

i	$K_2 \cap \ell_i$	$\beta_6 \cap \ell_i$
1	{9}	{2,16,23,30,37,44,51}
2	{9}	{1,10,11,12,13,14,15}
3	{9,52}	{8,22,28,34,40,46}
4	{9,45}	{5,19,29,32,42,55}
5	{4,9}	{18,27,36,38,47,56}
6	{9}	{7,21,26,31,43,48,53}
7	{6,9}	{20,24,35,39,50,57}
8	{9,49}	{3,17,25,33,41,57}
9	{4,6}	{1,2,3,5,7,8}
10	ϕ	{2,15,22,29,36,43,50,57}
11	ϕ	{2,12,19,26,33,40,47,54
12	ϕ	{2,11,18,25,32,39,46,53}
13	{49}	{2,14,21,28,35,42,56}
14	ϕ	{2,13,20,27,34,41,48,55}

15	[45,52}	{2,10,17,24,31,38}
16	{52,57}	{1,51,53,54,55,56}
17	{45}	{8,15,21,27,33,39,51}
18	ϕ	{5,12,22,25,35,38,48,51}
19	{4,49}	{11,20,29,31,40,51}
20	ϕ	{7,14,19,24,36,41,46,51}
21	{6}	{13,17,28,32,43,47,51}
22	ϕ	{3,10,18,26,34,42,50,51}
23	ϕ	{1,30,31,32,33,34,35,36}
24	{52}	{7,15,20,25,30,42,47}
25	{49}	{8,12,18,24,30,43,55}
26	{6,45}	{11,22,26,30,41,56}
27	ϕ	{5,14,17,27,30,40,50,53}
28	ϕ	{3,13,21,29,30,38,46,54}
29	{4}	{10,19,28,30,39,48,57}
30	ϕ	{1,23,24,25,26,27,28,29}
31	{6,49}	{19,15,23,34,38,53}
32	{4,52}	{12,21,23,32,41,50}
33	ϕ	{8,11,17,23,37,42,48,54}
34	ϕ	{3,14,22,23,31,39,47,55}
35	{45}	{7,13,18,23,35,40,57}
36	ϕ	{5,10,20,23,33,43,46,56}
37	{45,49}	{1,44,46,47,48,50}
38	ϕ	{5,15,18,28,31,41,44,54}
39	ϕ	{7,12,17,29,34,39,44,56}
40	{52}	{3,11,19,27,35,43,44}
41	ϕ	{8,14,20,26,32,38,44,57}
42	{4}	{13,22,24,33,42,44,53}
43	{6}	{10,21,25,36,40,44,55}
44	ϕ	{1,37,38,39,40,41,42,43}
45	{4}	{15,17,26,35,37,46,55}
46	{45}	{3,12,20,28,36,37,53}
47	ϕ	{5,11,21,24,34,37,47,57}
48	{6,52}	{14,18,29,33,37,48}
49	ϕ	{8,13,19,25,31,37,50,56}
50	{49}	{7,10,22,27,32,37,54}
51	ϕ	{1,16,17,18,19,20,21,22}
52	ϕ	{3,15,16,24,32,40,48,56}
53	{6}	{12,16,27,31,42,46,57}
54	ϕ	{7,11,16,28,33,38,50,55}
55	{4,45}	{14,16,25,34,43,54}
56	{49,52}	{5,13,16,26,36,39}
57	ϕ	{8,10,16,29,35,41,47,53}

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