i-Open Sets in Bitopological Spaces

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Received on: 27/06/2013
Accepted on: 11/11/2013

ABSTRACT

In this paper, we defined i-open sets and i-star generalized w-closed sets in
bitopological spaces \((X, \tau_1, \tau_2)\) by using the definition of i-open sets in topological
space \((X, \tau)\) (see[6]). We present some fundamental properties and relations between
these classes of sets, further we give examples to explain these relations.

Keywords: i-open sets, bitopological spaces.

المجامع المفتوحة من النوع–i في الفضاءات التبولوجية الثنائية

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تاريخ قبول البحث: 11/11/2013
تاريخ استلام البحث: 27/06/2013

الملخص

في هذا البحث، عرفنا المجاميع المفتوحة من النوع–i والمجمع المغلقة من النمط–i* ممممة من النوع-
w في الفضاءات التبولوجية الثنائية \((X, \tau_1, \tau_2)\) باستخدام تعريف المجاميع المفتوحة من النوع–i في الفضاء
التبولوجي \((X, \tau)\) (انظر [6]). تم إعطاء بعض الخصائص الأساسية والعلاقات بين هذه الأصناف من المجاميع
معززة بالأمثلة والبراهين.

الكلمات المفتاحية: المجاميع المفتوحة من النوع–i، الفضاءات التبولوجية الثنائية.

0. Introduction

Sheik and Sundaram in 2004 [9], introduced g* -closed sets in bitopological spaces. Kannan and Chandrasekhara in 2006 [4], introduced regular star generalized closed sets in bitopological spaces. Mahdi in 2007 [5], introduced the concept of semi-open and semi-closed sets in bitopological spaces. Benchalli, Patil and Rayanagoudar in 2010 [2], introduced w-locally closed sets in bitopological spaces. Sheik and Maragathavalli in 2010 [8], introduced the concept of strongly ag * – closed sets in bitopological spaces. Nagaveni and Rajarubi in 2012 [7], introduced GRW-closed sets and GRW-continuity in bitopological spaces. Mohammed and Askandar In 2012 [6], introduced the concept of i-open sets as: A subset \(A\) of a topological space \((X, \tau)\) is said to be i-open set[6] if there exists an open set \(G\neq \phi\), \(X\) such that \(A \subseteq Cl(A \cap G)\). The complement of an i-open set is called i-closed set, which could entire them together with many other concepts of generalized open sets. The aim of this paper is to introduce the concept of i-open sets in bitopological spaces \((X, \tau_1, \tau_2)\). This class of sets may be to enter together with other classes of sets in bitopological spaces which have been mentioned above for comparison and to find the similar properties and characterizations. Throughout this
work, \( \tau^i \) is a family of all i-open sets[6] of \( X \). This work consists of two sections. In the first one, we define i-open sets in bitopological spaces and we give many related examples. In the second section, we define i-star generalized w-closed sets, i-star generalized w-open sets and study their basic properties in bitopological spaces. \((X, \tau_i, \tau_j)\) denote a bitopological space, where \((X, \tau_i)\) and \((X, \tau_j)\) are topological spaces. For any subset \( A \subseteq X \), \( \tau_i - \text{Int}(A) \) and \( \tau_j - \text{Cl}(A) \) denote the interior and closure of a set \( A \) with respect to the topology \( \tau_i \). A point \( x \in X \) is called a condensation point of \( A \) [3] if for each \( U \in \tau \) with \( x \in U \), the set \( U \cap A \) is uncountable. A is called w-closed [3] if it contains all its condensation points. The complement of an w-closed set is called w-open. The w-closure [3] and w-\text{interior} [3] of \( A \) with respect to the topology \( \tau_i \), that can be defined in a manner similar to \( \tau_i - \text{Cl}(A) \) and \( \tau_i - \text{int}(A) \), respectively, will be denoted by \( \tau_i - \text{Cl}_w(A) \) and \( \tau_i - \text{int}_w(A) \), respectively. \( A^C \) denotes the complement of \( A \) in \( X \).

1. i-Open Sets in Bitopological Spaces.

In this section, we define i-open sets in bitopological spaces by giving many related examples and we study the properties of these sets. Also we define many concepts of generalized open sets in bitopological spaces and we give many related examples.

**Definition 1.1.** Let \((X, \tau_i, \tau_j)\) be a bitopological space, a subset \( A \) of \( X \) is said to be \((\tau_i, \tau_j - i - \text{open set})\) if there exists \( \tau_i - \text{open set} \ U \neq \phi, X \) s.t. \( A \subseteq \tau_j - \text{Cl}(A \cap U) \). The complement of \((\tau_i, \tau_j - i - \text{open set})\) is called \((\tau_i, \tau_j - i - \text{closed set})\).

**Definition 1.2.** A bitopological space \((X, \tau_i, \tau_j)\) is said to be Bi-Topologically Extended for i-open sets (Bi.T.E.I.) if \((X, \tau_i \tau_j - i - \text{open sets})\) is a topological space. On the other hand, if \((X, \tau_i, \tau_j - i - \text{open sets})\) is not a topological space then, \((X, \tau_i, \tau_j)\) is called non-Bi-Topologically Extended for i-open sets (not Bi.T.E.I.). Where, \( \tau_i \tau_j - i - \text{open sets} \) denote the family of all i-open sets in the bitopological space \((X, \tau_i, \tau_j)\).

**Example 1.3.** Let \( X = \{a, b, c\} \), \( \tau_1 = \{\phi, \{a\}, X\} \), \( \tau_2 = \{\phi, \{a\}, \{a, b\}, X\} \).

\( \tau_1 \)-open sets are: \( \phi, \{a\}, X \).
\( \tau_2 \)-closed sets are: \( \phi, \{b, c\}, \{c\}, X \).

\( \{a\} \subset (\tau_2 - \text{Cl}(\{a\} \cap \{a\})) = X \), \( \{a, b\} \subset (\tau_2 - \text{Cl}(\{a, b\} \cap \{a\})) = X \).

Then, \( \{a\}, \{a, b\}, \{a, c\} \) are \( \tau_i \tau_j - i - \text{open sets} \).

But, \( \{b\}, \{c\}, \{b, c\} \) are not \( \tau_i \tau_j - i - \text{open sets} \) because there is no existence \( \tau_1 - \text{open set} U \) s.t. \( \{b\} \subset (\tau_2 - \text{Cl}(\{b\} \cap U)), \{c\} \subset (\tau_2 - \text{Cl}(\{c\} \cap U)) \).

\( \{b, c\} \subset (\tau_2 - \text{Cl}(\{b, c\} \cap U)) \) Therefore, \( \tau_i \tau_j - i - \text{open set} = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\} \).

Where, \( \tau_i \tau_j - i - \text{closed set} = \{\phi, \{b, c\}, \{c\}, \{b\}, X \} \)

Therefore, \( (X, \tau_i \tau_j - i - \text{open set}) \) is a topological space. Then, \((X, \tau_i, \tau_j)\) is a Bi.T.E.I. space.

**Example 1.4.** Let \( X = \{a, b, c, d\} \), \( \tau_1 = \{\phi, \{a\}, \{b, c, d\} X\} \), \( \tau_2 = \{\phi, \{a\}, \{c\}, \{a, c\}, X\} \).
\( \tau_1 \) - open sets are: \( \phi, \{a\}, \{b, c, d\}, X \).
\( \tau_2 \) - closed sets are: \( \phi, \{b, c, d\}, \{a, b, d\}, \{b, d\}, X \).

By the same way, in Example 1.3, we have:
\( \tau_1 \tau_2 - i - open \) sets are:
\( \{\phi, \{a\}, \{b, c, d\}, \{d\}, \{b, c, d\}, \{b, d\}, \{e, c, d\}, \{b, c, d\}, \{b, d\}, \{a, b, d\}, \{b, d\}, X \} \)
\( \{b, c, d\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X \} \)
\( \{a, c\}, \{a, b\}, \{a, c, d\}, \{b, c\}, \{c, d\}, \{b, c, d\}, \{c, d\}, \{b, c, d\}, \{b, c, d\}, \{b, c, d\}, \{b, d\}, X \} \).

Where, \( (X, \tau_1 \tau_2 - i - open \) sets) \) is not a topological space. Then, \( (X, \tau_1, \tau_2) \) is not Bi.T.E.I. space.

**Definition 1.5.** Let \( (X, \tau') \) be a topological space and let \( A \) be a subset of \( X \). Recall that the intersection of all i-closed sets containing \( A \) is called i-closure of \( A \) [6], denoted by \( Cl_i(A) \): \( Cl_i(A) = \bigcap_{i\in I} F_i, A \subseteq F_i \ \forall i \) where, \( F_i \) is i-closed set \( \forall i \) in a topological space \( (X, \tau') \). \( Cl_i(A) \) is the smallest i-closed set containing \( A \).

**Definition 1.6.** Let \( (X, \tau') \) be a topological space and let \( A \) be a subset of \( X \). Recall that the union of all i-open sets contained in \( A \) is called i-Interior of \( A \) [6], denoted by \( Int_i(A) \). \( Int_i(A) = \bigcup_{i\in I} I_i \subseteq A \ \forall i \). Where, \( I_i \) is i-open set \( \forall i \) in a topological space \( (X, \tau') \). \( Int_i(A) \) is the largest i-open set contained in \( A \).

**Theorem 1.7.** Every \( \tau_1 - open \) set is \( (X, \tau_1, \tau_2) \) in \( i - open \) set
Or (\( \tau_1 \subseteq (\tau_1, \tau_2) - i - open \) sets)).

**Proof** Let \( X \) be a finite non empty set. Let \( \tau_1 = \{\phi, A_1, A_2, \ldots, A_n, X\} \), \( \tau_2 = \{\phi, B_1, B_2, \ldots, B_n, X\} \).

Where, \( A_i \subseteq X, B_i \subseteq X \ \forall i \).
\( \tau_1 - open \) sets are: \( \phi, A_1, A_2, \ldots, A_n, X \).
\( \tau_2 - closed \) sets are: \( \phi, X - B_1, X - B_2, \ldots, X - B_n, X \).
\( \tau_2 - Cl(A_i \cap A_j) = \bigcap_{A_i \cap A_j = F} F \), where \( F \) is \( \tau_2 - closed \) set.

At least, \( X \) is a \( \tau_2 - closed \) set contains \( A_i \cap A_j \ \forall i \).

Hence, \( \tau_2 - Cl(A_i \cap A_j) = \bigcap_{A_i \cap A_j = F} F = X \).

Therefore, \( A_i \subseteq (\tau_1, \tau_2) - i - open \) set.

Then, (\( \tau_1 \subseteq (\tau_1, \tau_2) - i - open \) sets)).
The converse of Theorem 1.7 is not true. Indeed, in Example 1.4 \{b, c\} is \( \tau_1 \tau_2 - i - open \) set, but is not \( \tau_1 - open \) set.

**Definition 1.8.** Let \( (X, \tau) \) be a topological space, recall that extension \( \tau' \) [6] is the family of all i-open subsets of space \( X \).

**Remark 1.9.** [6] \( (X, \tau) \) need not to be a topological space.

**Definition 1.10.** [6] A topological space \( (X, \tau) \) is said to be Topologically Extended for i-open sets (shortly T.E.I) if and only if \( (X, \tau) \) is a topological space. Otherwise, it is called not T.E.I.
Theorem 1.11. [6] Let $X$ be a non-empty finite set and let $\tau = \{\phi, A, X\}$ where, $A$ is a subset of $X$ and containing only one element. Then, $(X, \tau)$ is T.E.I. (i.e. $(X, \tau')$ is a topological space).

Corollary 1.12. Let $(X, \tau_1, \tau_2)$ be a bitopological space and let $(X, \tau_1)$ be a (T.E.I.) topological space as like as in Theorem 1.11, let $\tau_2 = \tau_1'$ where, $\tau_1'$ is the family of all $i$-open sets in a topological space $(X, \tau_1)$, then, $\tau_2 = \tau_1 - i$-open sets $\tau_1 \tau_2$

Proof Suppose that $X=\{x_1,x_2,\ldots,x_n\}$ and $\tau_1 = \{\phi, \{x_i\}, X\}.$

$i$-open sets are: $\phi, \{x_i\}, X.$

By definition of $i$-open sets, we have:

$\tau_1' = \{ \phi, \{x_1\}, \{x_1,x_2\}, \{x_1,x_2,x_3\}, \ldots, \{x_1,x_2,x_3,x_4\}, \ldots, \{x_1,x_2,x_3,x_4,\ldots,x_n\} \}.$

Since, $\tau_2 = \tau_1'$ then $\tau_2 - i$-closed sets are: $\{x_1,x_2,\ldots,x_n\} = X,$

$\{x_1,x_2,x_3,\ldots,x_n\}, \{x_1,x_2,x_3,\ldots,x_n\}, \{x_1,x_2,x_3,\ldots,x_n\}, \ldots, \{x_1,x_2,\ldots,x_n\}.$

Since, $\{x_1\}$ is the alone $i$-open set $\neq \phi, X$ and the intersection between $\{x_1\}$ and the sets $\{x_2\}, \{x_3\}, \ldots, \{x_n\}, \{x_2,x_3\}, \ldots, \{x_2,x_3,x_4\}, \ldots, \{x_2,x_3,x_4,\ldots,x_n\}, \ldots, \{x_2,x_3,\ldots,x_n\} \}.$

Example 1.13. Let $X = \{a,b,c\},$

$\tau_1 = \{\phi, \{a\}, X\}, \tau_2 = \tau_1' = \{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$

$\tau_2 - i$-closed sets are: $\phi, \{b,c\}, \{c\}, \{b\}, X.$

By the same way of the examples mentioned above, we have:

$\tau_1 \tau_2 - i$-open sets $= \tau_2$

Definition 1.14. A set $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called:

1. $\tau_1 \tau_2 - i$-generalized closed set $(\tau_1 \tau_2 - g$-closed set) [3] if $\tau_2 - Cl(A) \subseteq U$ where $A \subseteq U$ and $U \subseteq X$ is $\tau_1 - i$-open set.$$

2. $\tau_1 \tau_2 - g$-open set [3] if $X - A$ is $\tau_1 \tau_2 - g$-closed.

3. $\tau_1 - closed set.$ is $\subseteq X F \subseteq A$ where $F \subseteq \tau_2 - Int_i(A)$ if $\tau_1 \tau_2 - g$-open set

4. $\tau_1 \tau_2 - g i - open.$ is $X - A$ if $\tau_1 \tau_2 - g i - closed.$

5. $\tau_1 \tau_2 - i$-star genralzed closed set $(\tau_1 \tau_2 - i$* $g$-closed set) if $\tau_2 - Cl(A) \subseteq U$ where $A \subseteq U$ and $U \subseteq X$ is $\tau_1 - i$-open set. $\tau_1 - i$-star genralzed open set $(\tau_1 \tau_2 - i$* $g$-open set) if $X - A$ is $\tau_1 \tau_2 - i$* $g$-closed.

6. $\tau_1 \tau_2 - i$-star genralzed w-closed set $(\tau_1 \tau_2 - gw$-closed set )[1] if $\tau_2 - Cl_w(A) \subseteq U$ where $A \subseteq U$ and $U \subseteq X$ is $\tau_1 - open set.$

7. $\tau_1 \tau_2 - genralzed w$-open set $(\tau_1 \tau_2 - gw$-open set )[1] if $X - A$ is $\tau_1 \tau_2 - gw$-closed.

8. $\tau_1 \tau_2 - genralzed w$-open set $(\tau_1 \tau_2 - gw$-open set )[1] if $X - A$ is $\tau_1 \tau_2 - gw$-closed.
In the following example X is a finite set.

**Example 1.15.** Let \( X = \{a, b, c\} \), \( \tau_1 = \{\phi, \{a\}, X\} \) \( \tau_2 = \{\phi, \{a\}, X\} \).

From definitions mentioned above we have:

- \( \tau_1 \) - open sets: \( \phi, \{a\}, X \), \( \tau_1 \) - closed sets: \( \phi, \{b, c\}, X \).
- \( \tau_1 \) - w - closed sets: \( \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\} \), \( X \).
- \( \tau_1 \) - w - open sets: \( \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\} \), \( X \).
- \( \tau_1 \) - i - open sets: \( \phi, \{a\}, \{a, b\}, \{a, c\}, X \).
- \( \tau_1 \) - i - closed sets: \( \phi, \{b, c\}, \{c\}, \{b\} \), \( X \).
- \( \tau_2 \) - open sets: \( \phi, \{a\}, X \), \( \tau_2 \) - closed sets: \( \phi, \{b, c\}, X \).
- \( \tau_2 \) - w - closed sets: \( \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\} \), \( X \).
- \( \tau_2 \) - w - open sets: \( \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\} \), \( X \).
- \( \tau_2 \) - i - open sets: \( \phi, \{a\}, \{a, b\}, \{a, c\}, X \).
- \( \tau_2 \) - i - closed sets: \( \phi, \{b, c\}, \{c\}, \{b\} \), \( X \).

\{a\} is not \( \tau_1 \tau_2 \) - g - closed set because \( \tau_2 - cl(\{a\}) = X \subseteq X \)
but \( \tau_2 - cl(\{a\}) = X \subset \{a\} \) (definition(1.14)(1))

\( \tau_1 \tau_2 \) - g - open sets: \( \phi, \{a\}, \{a, b\}, \{c\}, \{b\}, \{a\}, X \).

but \( \{b, c\} \) is not \( \tau_1 \tau_2 \) - g - open set because \( \{a\}^C = \{b, c\} \)
and \( \{a\} \) is not \( \tau_1 \tau_2 \) - g - closed set (definition1.14(2))

\( \tau_1 \tau_2 - gi \) - open sets: \( \phi, X \), \( \tau_1 \tau_2 - gi \) - closed sets: \( \phi, X \).

\( \tau_1 \tau_2 - i* \) g - closed sets: \( \phi, \{b, c\}, X \), \( \tau_1 \tau_2 - i* \) g - open sets: \( \phi, \{a\}, X \).

\( \tau_1 \tau_2 - gw \) - closed sets: \( \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X \).

\( \tau_1 \tau_2 - gw \) - open sets: \( \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X \).

In the following example X is an infinite set.

**Example 1.16.** Let \( X = R \), \( \tau_1 = \{\phi, R \setminus Q, R\} \) \( \tau_2 = \{\phi, Q, R\} \). Where, \( R \) is the set of real numbers, \( Q \) is the set of rational numbers and \( R \setminus Q \) is the set of irrational numbers.

From definitions mentioned above we have:

- \( \tau_1 \) - open sets: \( \phi, R \setminus Q, R \), \( \tau_1 \) - closed sets: \( \phi, Q, R \).
- \( \tau_1 \) - w - closed sets: \( \phi, R \setminus Q, Q, R \), other sets \( \subseteq R \) which satisfies the definition of \( w \) - closed sets.
- \( \tau_1 \) - w - open sets: \( \phi, R \setminus Q, Q, R \), other sets \( \subseteq R \) which are the complements of \( \tau_1 \) - w - closed sets.
- \( \tau_1 \) - i - open sets: \( \phi, R \setminus Q, R \), other sets \( \subseteq R \) which it satisfies the definition of \( i \) - open sets. \( Q \) is not \( \tau_1 \) - i - open set.
- \( \tau_1 \) - i - closed sets: \( \phi, Q, R \), other sets \( \subseteq R \) which are the complements of \( \tau_1 \) - i - open sets. \( R \setminus Q \) is not \( \tau_1 \) - i - closed set.
- \( \tau_2 \) - open sets: \( \phi, Q, R \), \( \tau_2 \) - closed sets: \( \phi, R \setminus Q, R \).
- \( \tau_2 \) - w - closed sets: \( \phi, R \setminus Q, Q, R \), other sets \( \subseteq R \) which it satisfies the definition of \( w \) - closed sets.
\( \tau_2 \) - w - open sets: \( \phi, R - Q, Q, R, \) other sets \( \subseteq R \) which are the complements of \( \tau_2 \) - w - closed sets.
\( \tau_2 \) - i - open sets: \( \phi, Q, R, \) other sets \( \subseteq R \) which it satisfies the definition of \( i - \) open sets.
\( \tau_2 \) - i - closed sets: \( \phi, R - Q, R, \) other sets \( \subseteq R \) which are the complements of \( \tau_2 \) - i - open sets.
\( \tau_1 \tau_2 \) - g - closed sets: \( \phi, R - Q, Q, R, \) other sets \( \subseteq R \) which it satisfies the definition of \( \tau_1 \tau_2 \) - g - closed sets.
\( \tau_1 \tau_2 \) - g - open sets: \( \phi, R - Q, Q, R, \) other sets \( \subseteq R \) which are the complements of \( \tau_1 \tau_2 \) - g - closed sets.
\( \tau_1 \tau_2 \) - gi - open sets: \( \phi, Q, R, \) other sets \( \subseteq R \) which it satisfies the definition of \( \tau_1 \tau_2 \) - gi - open sets. \( R - Q \) is not \( \tau_1 \tau_2 \) - gi - open set.
\( \tau_1 \tau_2 \) - gi - closed sets: \( \phi, R - Q, Q, R, \) other sets \( \subseteq R \) which are the complements of \( \tau_1 \tau_2 \) - gi - open sets. \( Q \) is not \( \tau_1 \tau_2 \) - gi - closed set.
\( \tau_1 \tau_2 \) - i * g - closed sets: \( \phi, R - Q, Q, R, \) other sets \( \subseteq R \) which it satisfies the definition of \( \tau_1 \tau_2 \) - i * g - closed sets.
\( \tau_1 \tau_2 \) - i * g - open sets: \( \phi, R - Q, Q, R, \) other sets \( \subseteq R \) which are the complements of \( \tau_1 \tau_2 \) - i * g - closed sets.
\( \tau_1 \tau_2 \) - gw - closed sets: \( \phi, R - Q, Q, R, \) other sets \( \subseteq R \) which it satisfies the definition of \( \tau_1 \tau_2 \) - gw - closed sets.
\( \tau_1 \tau_2 \) - gw - open sets: \( \phi, R - Q, Q, R, \) other sets \( \subseteq R \) which are the complements of \( \tau_1 \tau_2 \) - gw - closed sets.

2. i-Star Generalized w-Closed and i-Star Generalized w-Open Sets in Bitopological Spaces.

Throughout this section, we define i-star generalized w-closed, i-star generalized w-open sets and study their basic properties in bitopological spaces.

**Definition 2.1.** A set \( A \) of a bitopological space \( (X, \tau_1, \tau_2) \) is said to be \( \tau_1 \tau_2 \) - i - star genralized w - closed set \( (\tau_1 \tau_2 \) - i * gw - closed set), if \( \tau_2 \) - Clw \( (A) \) \( \subseteq U \) where, \( A \subseteq U \) and \( U \subseteq X \) is a \( \tau_1 \) - i - open set.

In Example 1.15, we have:
\( \tau_1 \tau_2 \) - i * gw - closed sets: \( \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X \).

In Example 1.16, we have:
\( \tau_1 \tau_2 \) - i * gw - closed sets: \( \phi, R - Q, Q, R, \) other sets \( \subseteq R \) which it satisfies the definition of \( \tau_1 \tau_2 \) - i * gw - closed sets.

**Remark 2.2.** [6] Every open set in a topological space \( (X, \tau) \) is i-open.

**Theorem 2.3.** Let \( (X, \tau_1, \tau_2) \) be a bitopological space and \( A \subseteq X \) then, the followings are true:
1. If \( A \) is \( \tau_2 \) - w - closed then, \( A \) is \( \tau_1 \tau_2 \) - i * gw - closed.
2. If $A$ is $\tau_1 - i - open$ and $\tau_1 \tau_2 - i* gw - closed$ then, $A$ is $\tau_2 - w - closed$.
3. If $A$ is $\tau_1 \tau_2 - i* gw - closed$ then, $A$ is $\tau_1 \tau_2 - gw - closed$.

Proof
1. Suppose that $A$ is $i - open$, $\tau_1 - are$ $U \subseteq X$ and $A \subseteq U$. Let $\tau_2 - w - closed$ then $\tau_2 - Cl_w(A) = A \subseteq U$.
   Therefore, $A$ is $\tau_1 \tau_2 - i* gw - closed$.
2. Suppose that $A$ is $\tau_1 - i - open$ and $\tau_1 \tau_2 - i* gw - closed$. Let $A \subseteq A$ and $A$ is $\tau_1 - i - open$. Then, $\tau_2 - Cl_w(A) \subseteq A$. Therefore, $\tau_2 - Cl_w(A) = A$. Then, $A$ is $\tau_2 - w - closed$.
3. Suppose that $A$ is $\tau_1 \tau_2 - i* gw - closed$. Let $A \subseteq U$ and. Since, $\tau_1 - opens$ $U \subseteq X$ is $A$. Then, $\tau_2 - Cl_w(A) \subseteq U$ (Remark 2.2), we have $X$ in $i - open$ $\tau_1 - is U$.

$\blacksquare$

Theorem 2.4. Let $(X, \tau_1, \tau_2)$ be a bitopological space, then every $\tau_1 \tau_2 - i* gw - closed$ set in $X$ is $\tau_1 \tau_2 - i* gw - closed$.

Proof Suppose that $A$ is $\tau_1 \tau_2 - i* gw - closed$ set, we have $\tau_2 - Cl(A) \subseteq U$ , where $A \subseteq U$ and $U \subseteq X$ are $\tau_1 - i - open$ set.
Since, $\tau_2 - Cl_w(A) \subseteq \tau_2 - Cl(A)$,
we have $\tau_2 - Cl_w(A) \subseteq \tau_2 - Cl(A) \subseteq U$.
Therefore, $A$ is $\tau_1 \tau_2 - i* gw - closed$.

Remark 2.5. The converse of Theorem 2.4 is not true. Indeed, in Example 1.15, $A = \{a, b\}$ is $\tau_1 \tau_2 - i* gw - closed$ set, but is not $\tau_1 \tau_2 - i* g - closed$.

$\tau_1 \tau_2 - i* gw - closed$ set $\tau_1 \tau_2 - i* gw - closed$

Theorem 2.6. If $A$ is $\tau_1 \tau_2 - i* gw - closed$ set in $X$ and $A \subseteq B \subseteq \tau_2 - Cl_w(A)$, then $B$ is $\tau_1 \tau_2 - i* gw - closed$ set.

Proof Suppose that $A$ is $\tau_1 \tau_2 - i* gw - closed$ set in $X$ and $A \subseteq B \subseteq \tau_2 - Cl_w(A)$ . Let $B \subseteq U$ and $U$ is $\tau_1 - i - open$ set. Then, $A \subseteq U$ . Since, $A$ is $\tau_1 \tau_2 - i* gw - closed$ set,we have $\tau_2 - Cl_w(A) \subseteq U$ . Since, $B \subseteq \tau_2 - Cl_w(A)$, $\tau_2 - Cl_w(B) \subseteq \tau_2 - Cl_w(A) \subseteq U$ . Hence, $B$ is $\tau_1 \tau_2 - i* gw - closed$.

Theorem 2.7. If $A$ and $B$ are $\tau_1 \tau_2 - i* gw - closed$ sets then, so is $A \cup B$.

Proof Suppose that $A$ and $B$ are $\tau_1 \tau_2 - i* gw - closed$ sets . Let $U \subseteq X$ be $\tau_1 - i - open$ set and $(A \cup B) \subseteq U$ . Then, $A \subseteq U$ and $B \subseteq U$ . Since, $A$ and $B$ are $\tau_1 \tau_2 - i* gw - closed$ sets , we have $\tau_2 - Cl_w(A) \subseteq U$ and $\tau_2 - Cl_w(B) \subseteq U$ . Therefore, $\tau_2 - Cl_w(A \cup B) \subseteq U$ . Therefore, $A \cup B$ is $\tau_1 \tau_2 - i* gw - closed$ set.

Theorem 2.8. Let $(X, \tau_1, \tau_2)$ be a bitopological space and $A \subseteq X$ then, the following are true:
1. If $A$ is $\tau_2$-closed then, $A$ is $\tau_2$-w-closed.
2. If $A$ is $\tau_1$-$i$-$g$-closed then, $\tau_1\tau_2-g$-closed is $A$.
3. If $A$ is $\tau_1\tau_2-g$-closed then, $A$ is $\tau_1\tau_2-gw$-closed.

**Proof**

1. Suppose that $A$ is $\tau_2$-closed. Then $\tau_2-Cl(A)=A$. Since, $\tau_2-Cl(A)\subseteq\tau_2-Cl(A)=A$, we have $\tau_2-Cl(A)=A$. Therefore, $A$ is $\tau_2$-w-closed.
2. Suppose that $A$ is $A\subseteq U$. Let $\tau_1\tau_2-i^*g$-closed and $\tau_1$-open is $U\subseteq X$. Therefore, $\tau_2-Cl(A)\subseteq U$. Then, $A$ is $\tau_1\tau_2-g$-closed.
3. Suppose that $A$ is $\tau_1\tau_2-g$-closed. Let $A\subseteq U$ and $\tau_1$-open are $\tau_2-Cl(A)\subseteq U$. Then, $A$ is $\tau_1\tau_2-gw$-closed.

**Remark 2.9.** The converses of Theorem 2.8 are not true. Indeed, In Example 1.15, $A={a, c}$ is $\tau_2$-w-closed set, but is not $\tau_2$-closed, $A={a, c}$ is $\tau_1\tau_2$-g-closed set, but is not $\tau_1\tau_2-i^*g$-closed set. Also $\{a\}$ is $\tau_1\tau_2$-gw-closed set but, is not $\tau_1\tau_2-g$-closed.

$$\tau_2\text{-closed set } \tau_2\text{-w-closed}$$

$$\tau_1\tau_2-i^*g\text{-closed set } \tau_1\tau_2\text{-g-closed}$$

$$\tau_1\tau_2-g\text{-closed set } \tau_1\tau_2\text{-gw-closed}$$

**Definition 2.10.** A set $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $\tau_1\tau_2-i^*$-star genralized $w$-open set ($\tau_1\tau_2-i^*$ gw-open set), if $X-A$ is $\tau_1\tau_2-i^*$ gw-closed set.

In Example 1.15, we have:
$\tau_1\tau_2-i^*$ gw-open sets: $\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X$.

Also, in Example 1.16 we have:
$\tau_1\tau_2-i^*$ gw-open sets: $\phi, R-Q, Q, R, \text{other sets } \subseteq R\text{ which it satisfies the definition of } \tau_1\tau_2-i^*$ gw-open sets.

**Theorem 2.11.** A set $A$ is $\tau_1\tau_2-i^*$ gw-open set if and only if $F\subseteq\tau_2-Int(A)$, where $F\subseteq A$ and $F\subseteq X$ is $i$-closed set $\tau_1-$

**Proof** Suppose that $A$ is $\tau_1\tau_2-i^*$ gw-open set. Suppose that $F\subseteq X$ is $\tau_1-F^C$. Then $F\subseteq A$ and $i$-closed set is $A^C$. Since, $A^C\subseteq F^C$ and $i$-open $\tau_1-$
. Since, \( \tau_2 - Cl_w(A^C) \subseteq F^C \), we have \( \tau_1 \tau_2 - i^* gw - closed \).

Conversely, suppose that \( F \subseteq \tau_2 - Int_w(A) \) where \( F \subseteq A \) and \( F \subseteq X \) is \( \tau_1 - F^C \) and \( A^C \subseteq F^C \). Then, \( i - closed \) set is and \( F \subseteq \tau_2 - Int_w(A) \). Since, \( i - open \) \( \tau_1 - \) is \( A^C \). Then, \( \tau_2 - Cl_w(A^C) \subseteq F^C \), we have \( \tau_2 - Cl_w(A^C) = [\tau_2 - Int_w(A)]^F \).

Therefore, \( \tau_1 \tau_2 - i^* gw - closed \). Therefore, \( \tau_1 \tau_2 - i^* gw - closed \).

\textbf{Theorem 2.12.} If \( A \) and \( B \) are separated \( \tau_1 \tau_2 - i^* gw - open sets \), then so is \( A \cup B \).

\textbf{Proof} Suppose that \( A \) and \( B \) are \( \tau_1 \tau_2 - i^* gw - open sets \). Let \( F \subseteq X \) be \( \tau_1 - i - closed \) set and \( F \subseteq (A \cup B) \). Since \( A \) and \( B \) are separated sets, we have \( \tau_1 - Cl(A) \cap B = A \cap \tau_1 - Cl(B) = \phi \).

Also, \( \tau_2 - Cl(A) \cap B = A \cap \tau_2 - Cl(B) = \phi \). Then, \( F \cap \tau_2 - Cl(A) \subseteq (A \cup B) \cap \tau_2 - Cl(A) = A \). By the same way, we have \( F \cap \tau_2 - Cl(B) \subseteq B \). Since, \( F \subseteq X \) is \( \tau_1 - i - closed \) set, we have \( F \cap \tau_1 - Cl(A) \) and \( F \cap \tau_1 - Cl(B) \) are \( \tau_1 - i - closed \) sets. Since, \( A \) and \( B \) are \( \tau_1 \tau_2 - i^* gw - open sets \), we have \( F \cap \tau_2 - Cl(A) \subseteq \tau_2 - Int_w(A) \) and \( F \cap \tau_2 - Cl(B) \subseteq \tau_2 - Int_w(B) \).

Now \( F = F \cap (A \cup B) \subseteq (F \cap \tau_2 - Cl(A)) \cup (F \cap \tau_2 - Cl(B)) \subseteq \tau_2 - Int_w(A \cup B) \).

Therefore, \( A \cup B \) is \( \tau_1 \tau_2 - i^* gw - open set \).

\textbf{Theorem 2.13.} If \( A \) and \( B \) are \( \tau_1 \tau_2 - i^* gw - open sets \) then so is \( A \cap B \).

\textbf{Proof} Suppose that \( A \) and \( B \) are \( \tau_1 \tau_2 - i^* gw - open sets \). Let \( F \subseteq X \) be \( \tau_1 - i - closed \) set and \( F \subseteq (A \cap B) \), we have \( F \subseteq A \) and \( F \subseteq B \). Since, \( A \) and \( B \) are \( \tau_1 \tau_2 - i^* gw - open sets \), we have \( F \subseteq \tau_2 - Int_w(A) \) and \( F \subseteq \tau_2 - Int_w(B) \). Then \( F \subseteq \tau_2 - Int_w(A \cap B) \).

Therefore, \( A \cap B \) is \( \tau_1 \tau_2 - i^* gw - open set \).

\textbf{Theorem 2.14.} If \( A \) is \( \tau_1 \tau_2 - i^* gw - open set \) in \( X \) and \( \tau_2 - Int_w(A) \subseteq B \subseteq A \), then \( B \) is \( \tau_1 \tau_2 - i^* gw - open set \).

\textbf{Proof} Suppose that \( A \) is \( \tau_1 \tau_2 - i^* gw - open set \) in \( X \) and \( \tau_2 - Int_w(A) \subseteq B \subseteq A \). Let \( F \subseteq X \) be \( \tau_1 - i - closed \) set and \( F \subseteq B \). Since, \( F \subseteq B \) and \( B \subseteq A \), we have \( F \subseteq A \). Since, \( A \) is \( \tau_1 \tau_2 - i^* gw - open set \), we have \( F \subseteq \tau_2 - Int_w(A) \) and since, \( \tau_2 - Int_w(A) \subseteq B \), we have \( \tau_2 - Int_w(A) \subseteq \tau_2 - Int_w(B) \). Then, \( F \subseteq \tau_2 - Int_w(B) \).

Therefore, \( B \) is \( \tau_1 \tau_2 - i^* gw - open \).

\textbf{Theorem 2.15.} Let \( (X, \tau_1, \tau_2) \) be a bitopological space and \( A \subseteq X \) then the followings are true:

1. If \( A \) is \( \tau_2 - w - open \), then \( A \) is \( \tau_1 \tau_2 - i^* gw - open \).
2. If \( A \) is \( \tau_1 - i - closed \) and \( \tau_1 \tau_2 - i^* gw - open \), then \( A \) is \( \tau_2 - w - open \).
3. If \( A \) is \( \tau_1 \tau_2 - i^* gw - open \) then \( A \) is \( \tau_1 \tau_2 - gw - open \).
4. If \( A \) is \( \tau_1\tau_2 - i^* \text{ open} \) then \( A \) is \( \tau_1\tau_2 - i^* \text{ gw-open} \).
5. If \( A \) is \( \tau_2 - \text{ open} \) then \( A \) is \( \tau_2 - \text{ w-open} \) [3].
6. If \( A \) is \( \tau_1\tau_2 - i^* \text{ gw-open} \) then \( A \) is \( \tau_1\tau_2 - g - \text{ open} \).
7. If \( A \) is \( \tau_1\tau_2 - g - \text{ open} \) then \( A \) is \( \tau_1\tau_2 - \text{ gw-open} \).

**Proof**

1. Suppose that \( A \) is \( \tau_2 - \text{ w-open} \). We have \( A^C \) is \( \tau_2 - \text{ w-closed} \). Then, \( A^C \) (Theorem 2.3(1)) \( \tau_1\tau_2 - i^* \text{ gw-closed} \).
   Therefore, \( A \) is \( \tau_1\tau_2 - i^* \text{ gw-open} \).

2. Suppose that \( A \) is \( \tau_1 - \text{ i-closed} \) and \( \tau_1\tau_2 - i^* \text{ gw-open} \). Then, \( A^C \) is \( \tau_1 - i^* \text{ open} \) and \( \tau_1\tau_2 - i^* \text{ gw-closed} \).
   Then, \( \tau_2 - \text{ w-open} \) is \( A \) (Theorem 2.3(2)). Therefore, \( \tau_2 - \text{ w-closed} \) is \( A^C \).

3. Suppose that \( A \) is \( \tau_1\tau_2 - i^* \text{ gw-open} \). Then, \( A^C \) is \( \tau_1\tau_2 - i^* \text{ gw-closed} \), hence \( A^C \) is \( \tau_1\tau_2 - \text{ gw-closed} \) (Theorem 2.3(3)). Therefore, \( A \) is \( \tau_1\tau_2 - \text{ gw-open} \).

4. Suppose that \( A \) is \( \tau_1\tau_2 - i^* \text{ g-open} \). Then, \( A^C \) is \( \tau_1\tau_2 - i^* \text{ g-closed} \), hence \( A^C \) is \( \tau_1\tau_2 - i^* \text{ gw-closed} \) (Theorem 2.4). Therefore, \( A \) is \( \tau_1\tau_2 - i^* \text{ gw-open} \).

5. (see [3]).

6. Suppose that \( A \) is \( A^C \), hence \( \tau_1\tau_2 - i^* \text{ g-closed} \) is \( A^C \). Then, \( \tau_1\tau_2 - i^* \text{ g-open} \) \( \tau_1\tau_2 - g - \text{ closed} \) (Theorem 2.8(2)).
   Therefore, \( A \) is \( \tau_1\tau_2 - g - \text{ open} \).

7. Suppose that \( A \) is \( \tau_1\tau_2 - g - \text{ open} \). Then, \( A^C \) is \( \tau_1\tau_2 - g - \text{ closed} \), hence \( A^C \) is \( \tau_1\tau_2 - \text{ gw-closed} \) (Theorem 2.8(3)).
   Therefore, \( A \) is \( \tau_1\tau_2 - \text{ gw-open} \).

**Remark 2.16.** The converses of Theorem 2.15(4)(5)(6)(7) are not true. Indeed, In Example 1.15, \( A = \{b\} \) is \( \tau_1\tau_2 - i^* \text{ gw-open} \), but it is not \( \tau_1\tau_2 - i^* \text{ g-open} \) \( \{b\} \) is \( \tau_2 - \text{ w-open set} \), but it is not \( \tau_2 - \text{ open} \). Also, \( A = \{b\} \) is \( \tau_1\tau_2 - g - \text{ open set} \), but it is not \( \tau_1\tau_2 - i^* \text{ g-open set} \).

\( \{b, c\} \) is \( \tau_1\tau_2 - \text{ gw-open set} \) but it is not \( \tau_1\tau_2 - g - \text{ open} \).
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