Nullity and Bounds to the Nullity of Dendrimer Graphs

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ABSTRACT

In this paper, a high zero-sum weighting is applied to evaluate the nullity of a
dendrimer graph for some special graphs such as cycles, paths, complete graphs,
complete bipartite graphs and star graphs.

Finally, we introduce and prove a sharp lower and a sharp upper bound for the
nullity of the coalescence graph of two graphs.

Keywords: Graph spectra, Nullity of graphs

1. Introduction

The characteristic polynomial of the adjacency matrix A(G) is said to be the
characteristic polynomial of the graph G, denoted by \( \varphi(G; x) \). The eigenvalues of A(G)
are said to be the eigenvalues of the graph G, the occurrence of zero as an eigenvalue
in the spectrum of the graph G is called the “nullity” of G denoted by \( \eta(G) \). Brown and
others [4] proved that a graph G is singular if, and only if, G possesses a non-trivial
zero-sum weighting, and asked, what causes a graph to be singular and what are the
effects of this on its properties. Rashid [11] proved that a high zero-sum weighting
\( M_v(G) \) of a graph G, that is (the maximum number of non zero independent variables
used in a high zero- sum weighting for a graph G, is equal to the nullity of G) It is
known that \( 0 \leq \eta(G) \leq p - 2 \) if G is a non empty graph with p vertices. Cheng and Liu [5]
proved that if G has p vertices with no isolated vertices, then \( \eta(G) = p - 2 \) if, and only if,
G is isomorphic to a complete bipartite graph \( K_{m,n} \), and \( \eta(G) = p - 3 \) if, and only if, G
is isomorphic to a complete 3 partite graph \( K_{a,b,c} \). Omidi [10] found some lower bounds
for the nullity of graphs and proved that among bipartite graphs with p vertices, q edges
and maximum degree \( \Delta \) which do not have any cycle of length a multiple of 4 as a
subgraph, the greatest nullity is \( p - 2 \left\lfloor q / \Delta \right\rfloor \).
In this paper, we continue the research along the same lines. We derive formulas to determine the nullity of dendrimer graphs.

2 Definition and Preliminary Results

**Definition 2.1**: [5, p.16] and [8] A vertex weighting of a graph G is a function \( f: V(G) \rightarrow \mathbb{R} \) where \( \mathbb{R} \) is the set of real numbers, which assigns a real number (weight) to each vertex. The weighting of G is said to be non-trivial if there is, at least, one vertex \( v \in V(G) \) for which \( f(v) \neq 0 \).

**Definition 2.2**: [5, p.16] A non-trivial vertex weighting of a graph G is called a zero-sum weighting provided that for each \( v \in V(G) \), \( \sum f(w) = 0 \), where the summation is taken over all \( w \in NG(v) \).

Clearly, the following weighting for G is a non-trivial zero-sum weighting where \( x_1, x_2, x_3, x_4, \) and \( x_5 \) are weights and provided that \( (x_1, x_2, x_3, x_4, x_5) \neq (0, 0, 0, 0, 0) \) as indicated in Figure 2.1.

**Figure 2.1.** A non-trivial zero-sum weighting for a graph G.

**Theorem 2.3**: [4] a graph g is singular if, and only if, there is a non-trivial zero-sum weighting for g.■

Hence, the graph G depicted in Figure 2.1 is singular. Out of all zero-sum weightings of a graph G, a high zero-sum weighting of G is one that uses maximum number of non-zero independent variables.

**Proposition 2.4**: [6, p.35] and [8] in any graph g, the maximum number \( mv(g) \) of non-zero independent variables in a high zero-sum weighting equals the number of zeros as an eigenvalues of the adjacency matrix of g, (i.e. \( mv(g) = \eta(g) \)).■

In Figure 2.1, the weighting for the graph G is a high zero-sum weighting that uses 5 independent variables, hence, \( \eta(G) = 5 \).

The complement of the disjoint union of m edges is called a cocktail graph and is denoted by \( CP(m) = (mK_2)c = K_2,2,\ldots,2 = K_m(2) \).

**Proposition 2.5**: [6, p.20] The spectrum of the cocktail graph CP(m) is:

\[
S_p(CP(m)) = \begin{pmatrix} 2m-2 & 0 & -2 \\ 1 & m & m-1 \end{pmatrix}, \text{thus } \eta(CP(m)) = \begin{cases} 2, & \text{if } m=1, \\ m, & \text{if } m>1. \end{cases}
\]

**Proposition 2.6**: [2] The adjacency matrix of the wheel graph Wp, \( A(W_p) \), has eigenvalues \( 1+\sqrt{p}, 1-\sqrt{p}, \) and \( 2\cos \frac{2\pi r}{p-1}, r = 0, 1, \ldots, p-2 \). Hence, \( \eta(W_p) = 2 \) if \( p=1 \) (mod 4) and 0 otherwise.

**Proposition 2.7**: [4, p.72]i) The eigenvalues of the cycle Cp are of the form \( 2\cos \frac{2\pi r}{p}, \) \( r = 0, 1, \ldots, p-1 \). According to this, \( \eta(CP) = 2 \) if \( p=0 \) (mod 4) and 0 otherwise.

ii) The eigenvalues of the path Pp are of the form \( 2\cos \frac{\pi r}{p+1}, r =1,2, \ldots p \). And thus, \( \eta(P) = 1 \) if \( p \) is odd and 0 otherwise.

iii) The spectrum of the complete graph Kp, consists of \( p-1 \) and \(-1\) with multiplicity \( p-1 \).
Nullity and Bounds to the Nullity of Dendrimer Graphs

iv) The spectrum of the complete bipartite graph $K_{m,n}$, consists of $\sqrt{mn}$, $-\sqrt{mn}$ and zero $m+n$ - 2 times

**Corollary 2.8:** [4, p.234] If $G$ is a bipartite graph with an end vertex, and if $H$ is an induced subgraph of $G$ obtained by deleting this vertex together with the vertex adjacent to it, then $\eta(G) = \eta(H)$. ■

**Corollary 2.9:** [4, p.235] Let $G_1$ and $G_2$ be two bipartite graphs in which $\eta(G_1) = 0$. If the graph $G$ is obtained by joining an arbitrary vertex of $G_1$ by an edge to an arbitrary vertex of $G_2$, then $\eta(G) = \eta(G_2)$. ■

### Coalescence Graphs

To **identify** nonadjacent vertices $u$ and $v$ of a graph $G$ is to replace the two vertices by a single vertex incident to all the edges which are incident in $G$ to either $u$ or $v$.

Denote the resulting graph by $G\{u, v\}$. To **contract** an edge $e$ of a graph $G$ is to delete the edge and then (if the edge is a link) identify its ends. The resulting graph is denoted by $G/e$.

**Definition 2.10:** [7] Let $(G_1, u)$ and $(G_2, v)$ be two graphs rooted at vertices $u$ and $v$, respectively. We attach $G_1$ to $G_2$ (or $G_2$ to $G_1$) by identifying the vertex $u$ of $G_1$ with the vertex $v$ of $G_2$. Vertices $u$ and $v$ are called **vertices of attachment**. The vertex formed by their identification is called the **coalescence vertex**. The resulting graph $G_1 \circ G_2$ is called the **coalescence (vertex identification)** of $G_1$ and $G_2$.

**Definition 2.11:** [7] Let $\{(G_1, v_1), (G_2, v_2), \ldots, (G_t, v_t)\}$ be a family of not necessary distinct connected graphs with roots $v_1, v_2, \ldots, v_t$, respectively. A connected graph $G= G_1 \circ G_2 \circ \ldots \circ G_t$ is called the **multiple coalescence** of $G_1, G_2, \ldots, G_t$ provided that the vertices $v_1, v_2, \ldots, v_t$ are identified to reform the coalescence vertex $v$. The **$t$-tuple coalescence graph** is denoted by $G_t \circ G$. is the multiple coalescence of $t$ isomorphic copies of a graph $G$. In the same ways $G_1 \circ G_2$ is the multiple coalescence of $G_1$ and $t$ copies of $G_2$.

**Remark 2.12:** [7] All coalesced graphs have $v$ as a common cut vertex. Some graphs and their operations will, herein, be illustrated in Figure 2.2.

![Figure 2.2](image)

**Figure 2.2** Multiple coalescence $G_1 \circ G_2 \circ G_3$, $t$-tuple coalescence $G_t \circ G_1$ and coalescence of both $G_1 \circ G_2$.

**Definition 2.13:** [7] Let $G$ be a graph consisting of $n$ vertices and $L = \{H_1, H_2, \ldots, H_n\}$ be a family of rooted graphs. Then, the graph formed by attaching $H_k$ to the $k$-th $(1 \leq k \leq n)$ vertex of $G$ is called the **generalized rooted product** and is denoted by $G(L)$; $G$ itself is called the **core** of $G(L)$. If each member of $L$ is isomorphic to the rooted graph $H$, then the graph $G(L)$ is denoted by $G(H)$. Recall $G_1$, $G_2$ and $G_3$ from Figure 2.2. Then, we have
Definition 2.14: [7] The generalization of the rooted product graphs is called the F-graphs, which are consecutively iterated rooted products defined as: $\mathcal{F}_0 = K_1$, $\mathcal{F}_1 = G = H$, $\mathcal{F}_i = G(H)$, $\ldots$, $\mathcal{F}_s = \mathcal{F}^s(H)$, $s \geq 1$.

Definition 2.15: [7] A family of dendrimers $D_k$ ($k \geq 0$) is just a rooted product graph which is defined as follows:

- $D_0 = K_1$,
- $D_1 = G = H$,
- $D_{i+1} = \mathcal{F}^{i+1}(H)$, $i \geq 0$.

In general, $D_k$ ($k \geq 1$) is constructed from $D_{k-1}$, and the number of copies of $H$ attached to $D_k$ obeys some fixed generation law. Hence, $D_k$ is $D$ with $G$ attached to each vertex of $D$ which is not in $D_{k-1}$, that is to each $u \in V(D) - V(D_{k-1})$, $k \geq 1$.

3 Nullity of Dendrimer Graphs

In this section, we determine the nullity of dendrimer graphs $D_k^i$, $k \geq 0$, where $D^i = G^i$ of some known graphs such as $C_p$, $P_p$, $K_p$ and $K_{m,n}$. In each case, we consider that the nullity of the dendrimer graph $D^0$ is defined to be, $\eta(D^0) = \eta(K_t) = 1$. The dendrimer $C_p^k$ for the cycle $C_p$ is a connected graph with order $p(C_p^k) = p + p(p-1) + p(p-1)^2 + \ldots + p(p-1)^{k-1} = \sum_{i=1}^{k-1} P(p-1)^{i-1}$. And size $q(C_p^k) = q + pq + p(p-1)q + \ldots + p(p-1)^{k-2}q$.

Moreover, the diameter of $C_p^k$ is $(2k - 1)$, $\text{diam}(C_p^k)$. Also for $k > 1$, the degrees of each vertex of $C_p^k$ is either 2 or 4.
**Proposition 3.1:** For a dendrimer graph $C_p^k$, $k \geq 1$, we have:

i) If $p = 4n$, $n = 1, 2, \ldots$, then $\eta(C_{4n}^k) = 2$.

And for all $k$, $k \geq 2$, $\eta(C_{4n}^k) = \eta(C_{4n}^{k-1}) + 4n(4n-1)^{k-2}$.

ii) If $p = 4n + 2$, $n = 1, 2, \ldots$, then $\eta(C_{4n+2}^k) = 0$, for all $k$, $k \geq 1$.

iii) If $p = 4n - 1$, $n = 1, 2, \ldots$, then $\eta(C_{4n-1}^k) = 0$, $\eta(C_{4n-1}^2) = 1$.

And for all $k$, $k \geq 3$, $\eta(C_{4n-1}^k) = 0$.

iv) If $p = 4n + 1$, $n = 1, 2, \ldots$, then $\eta(C_{4n+1}^k) = 0$. for all $k$, $k \geq 1$.

**Proof:** i) For $k = 1$ it is clear that $\eta(C_{4n}^1) = 2$, $n = 1, 2, \ldots$, by Proposition 2.7 (i). For $k = 2$, $C_{4n}(C_{4n})$, is a rooted product of $C_{4n}$ and $C_{4n}$. So we need to prove that $\eta(C_{4n}^2) = 2 + 4n$. Let $x_{i,j}$, $i, j = 1, 2, \ldots, 4n$ be a weighting for the vertex $v_{i,j}$ in $C_{4n}^2$, $n = 1, 2, \ldots$, as indicated in Figure 3.1.

![Figure 3.1. A weighting of $C_{4n}^2$, $n = 1, 2, \ldots$](image)

From the condition that $\sum_{w \in N_G(v)} f(w) = 0$, for all $v$ in $C_{4n}^2$, $n = 1, 2, \ldots$, we have, for the cycles identified with the vertices $v_{i,j}$.

For $j = 1, 3, \ldots, 4n - 3$.

$\begin{align*}
x_{1,j} + x_{1,j+2} = 0 & \quad \Rightarrow \quad x_{1,j} = -x_{1,j+2} \\
x_{2,j} + x_{2,j+2} = 0 & \quad \Rightarrow \quad x_{2,j} = -x_{2,j+2} \\
\vdots & \\
x_{4n,j} + x_{4n,j+2} = 0 & \quad \Rightarrow \quad x_{4n,j} = -x_{4n,j+2}
\end{align*}$

...(3.1)

And, for $j = 2, 4, \ldots, 4n - 2$.

$\begin{align*}
x_{1,j} + x_{1,j+2} = 0 & \quad \Rightarrow \quad x_{1,j} = -x_{1,j+2} \\
x_{2,j} + x_{2,j+2} = 0 & \quad \Rightarrow \quad x_{2,j} = -x_{2,j+2} \\
\vdots & \\
x_{4n,j} + x_{4n,j+2} = 0 & \quad \Rightarrow \quad x_{4n,j} = -x_{4n,j+2}
\end{align*}$

...(3.2)

Also, from the condition that $\sum_{w \in N_G(v)} f(w) = 0$, for all $v$ in the central cycle $C_{4n}$, we have, for $i = 1, 3, \ldots, 4n - 3$.

$\begin{align*}
x_{i,1} + x_{i+2,1} = 0 & \quad \Rightarrow \quad x_{i,1} = -x_{i+2,1} \\
\end{align*}$

...(3.3)

And, for $i = 2, 4, \ldots, 4n - 2$.

$\begin{align*}
x_{i,1} + x_{i+2,1} = 0 & \quad \Rightarrow \quad x_{i,1} = -x_{i+2,1} \\
\end{align*}$

...(3.4)
Therefore, for each $i$ in the Equations (3.1), (3.2) and (3.4) we have used exactly two non-zero independent variables, one of which in the weight of $x_{i,1}$, where $i$ is odd and the other in the weight of $x_{i,1}$, where $i$ is even. And from Equation (3.2) we have used $4n$ non-zero independent variables.

Thus, the maximum number of non-zero independent variables used in a high zero-sum weighting of $C_{4n}^2$, $n = 1, 2, \ldots$, is equal to $2 + 4n$.

On the other hand, we have $\eta(C_{4n}) = 2$, $n = 1, 2, \ldots$, by Lemma 2.7 (i). But $C_{4n}^2 = C_{4n}(C_{4n})$, so each identification of a copy of $C_{4n}$ with a vertex of $C_{4n}$ adds (increases) one to the nullity of a dendrimer graph. Since $C_{4n}$ has $4n$ vertices; thus, $4n$ copies of a cycle $C_{4n}$ are identified to $C_{4n}$.

Therefore, $\eta(C_{4n}(C_{4n})) = \eta(C_{4n}) + 4n = 2 + 4n$.

For $k \geq 3$, we use the iteration $C_{4n}^2(C_{4n})$. This graph is a rooted product of $C_{4n}^2$ and $C_{4n}$. Since $C_{4n}^2$ is a dendrimer graph having $4n$ cycles and each cycle has $4n - 1$ vertices to be identified with new vertices, hence we attach a copy of $C_{4n}$ to $4n(4n - 1)$ vertices. Also, each copy of $C_{4n}$ adds (increases) one to the nullity of a dendrimer graph. Therefore, $\eta(C_{4n}^2(C_{4n})) = \eta(C_{4n}) + 4n(4n - 1) = 2 + 4n + 4n(4n - 1)$.

Similarly, we have, $\eta(C_{4n}^{k-1}(C_{4n})) = \eta(C_{4n}) + 4n(4n - 1)^{k-2}$ where $k \geq 3$.

ii) For each $k$, $k \geq 1$, there exists no non-trivial zero-sum weighting for $C_{4n+2}^k$, $n = 1, 2, \ldots$. Thus, by Theorem 2.3, $C_{4n+2}^k$ is non-singular.

iii) For $k = 1$, there exists no non-trivial zero-sum weighting for $C_{4n-1}^1$, $n = 1, 2, \ldots$. Thus, by Theorem 2.3, $C_{4n-1}^1$ is non-singular. For $k = 2$, $C_{4n-1}(C_{4n-1})$, is a rooted product of $C_{4n-1}$ and $C_{4n-1}$. To prove that $\eta(C_{4n-1}(C_{4n-1})) = 1$. Let $x_{i,j}$, $i, j = 1, 2, \ldots, 4n - 1$ be a weighting for

![Figure 3.2. A weighting of $C_{4n-1}^2$, $n = 1, 2, \ldots$](image)

vertex $y_{i,j}$ in $C_{4n-1}^2$, $n = 1, 2, \ldots$, as indicated in Figure 3.2.
Then, from the condition that \( \sum_{w \in \mathcal{N}_G(v)} f(w) = 0 \), for all \( v \in C_{4n-1}^2 \), \( n = 1, 2, \ldots, \) we have:

For \( i = 1, 2, \ldots, 4n-1 \), and \( j = 1, 3, \ldots, 4n-3 \).

\[
x_{i,j} + x_{i,j+2} = 0 \quad \Rightarrow \quad x_{i,j} = -x_{i,j+2} \quad \ldots (3.5)
\]

And, for \( i = 1, 2, \ldots, 4n-1 \), and \( j = 2, 4, \ldots, 4n-2 \).

\[
x_{i,j} + x_{i,j+2} = 0 \quad \Rightarrow \quad x_{i,j} = -x_{i,j+2} \quad \ldots (3.6)
\]

Hence, from Equations (3.5) and (3.6), we get:

\[
x_{1,1} = x_{1,4} = x_{1,5} = x_{1,8} = x_{1,9} = \ldots = x_{1,4n-4} = x_{1,4n-3} = -x_{i,2} \quad \ldots (3.7)
\]

And \( x_{1,2} = x_{1,3} = x_{1,6} = x_{1,7} = \ldots = x_{1,4n-2} = x_{1,4n-1} = -x_{1,1} \quad \ldots (3.8)\)

Also, from the condition that \( \sum_{w \in \mathcal{N}_G(v)} f(w) = 0 \), for all \( v \in C_{4n-1}^2 \), \( n = 1, 2, \ldots, \) we have:

\[
x_{1,2} + x_{1,4n-1} + x_{2,1} + x_{4n-1,1} = 0
\]

Since \( x_{1,2} = x_{1,4n-1} = -x_{1,1} \), therefore, \( x_{2,1} = x_{4n-1,1} = x_{1,1} \quad \ldots (3.9)\)

Hence, from Equations (3.7), (3.8) and (3.9), we get:

For \( i = 1, 2, \ldots, 4n-1 \).

\[
x_{i,1} = x_{i,4} = x_{i,5} = x_{i,8} = x_{i,9} = \ldots = x_{i,4n-4} = x_{i,4n-3} = x_{i,1} \quad \ldots (3.10)
\]

And, for \( i = 1, 2, \ldots, 4n-1 \).

\[
x_{i,2} = x_{i,3} = x_{i,6} = x_{i,7} = \ldots = x_{i,4n-2} = x_{i,4n-1} = -x_{i,1} \quad \ldots (3.11)
\]

Therefore, each vertex of \( C_{4n-1}^2 \), \( n = 1, 2, \ldots \) has a weight \( x_{i,1} \) or \( -x_{i,1} \).

This means that there exists a non-trivial zero-sum weighting for \( C_{4n-1}^2 \) used exactly one non-zero independent variable in a high zero-sum weighting of \( C_{4n-1}^2 \). Hence, \( \eta(C_{4n-1}^2) = 1 \).

Finally, the proof of \( \eta(C_{4n-1}^2) = 0 \), for \( k \geq 3 \), is similar to that for \( k=2 \).

**iv) The proof is similar to that of part (ii).**

**Corollary 3.2:** For a dendrimer graph \( C_{4n}^k \), \( k \geq 2 \), \( n = 1, 2, \ldots \), we have

\[
\eta(C_{4n}^k) = 2 + 4n \left[ \sum_{i=0}^{k-2} (4n-1)^i \right].
\]

**Proof:** From Proposition 3.1 (i), we have:

\[
\eta(C_{4n}^k) = \eta(C_{4n}^{k-1}) + 4n (4n-1)^{k-2}, \quad \text{for} \quad k \geq 2
\]

\[
\therefore \quad \eta(C_{4n}^k) = \eta(C_{4n}^{k-1}) + 4n (4n-1)^{k-3} + 4n (4n-1)^{k-2}
\]

\[
= \eta(C_{4n}^{k-1}) + 4n (4n-1)^{k-4} + 4n (4n-1)^{k-3} + 4n (4n-1)^{k-2}
\]

\[
\therefore \quad \eta(C_{4n}^k) = 2 + 4n \left[ \sum_{i=0}^{k-2} (4n-1)^i \right]
\]

\[
= 2 + 4n \left[ \sum_{i=0}^{k-2} (4n-1)^i \right], \quad \text{for} \quad k \geq 2.
\]
Let \( P_p \) be a path with usually labeled vertices \( v_1, v_2, \ldots, v_p \). If \( p \) is odd, this graph has a non-trivial zero-sum weighting, say \( x, 0, -x, 0, \ldots \), which provides that an odd path is singular. Moreover, the dendrimer \( P_p^k \) has order
\[
p(P_p^k) = p + p(p - 1) + p(p - 1)^2 + \ldots + p(p - 1)^{k-1}
\]
and size \( q(P_p^k) = p(P_p^k) - 1 \).

While, the diameter of \( D^k \) depends on the choice of the rooted vertex. Also, the maximum degree will be either 3 or 4 for \( k \geq 2 \), while the minimum degree is 1.

In general, \( diam(P_p^k) \leq (2k - 1)(p - 1) \), equality holds if \( k=1 \) or the rooted vertex is an end vertex of the path.

**Proposition 3.3:** For a dendrimer graph \( P_p^k, k \geq 1 \) we have:

i) If \( p = 2n, n = 1, 2, \ldots \), then \( \eta(P_p^k) = 0 \) for all \( k, k \geq 1 \).

ii) If \( p = 2n + 1, n = 1, 2, \ldots \), and the rooted vertex has a non-zero weight, then
\[
\eta(P_{2n+1}^k) = 1 \text{ for all } k, k \geq 1.
\]

iii) If \( p = 2n + 1, n = 1, 2, \ldots \), and the rooted vertex has a zero weight, then
\[
\eta(P_{2n+1}^k) = 1, \eta(P_{2n+1}^2) = 2n + 1, \text{ and}
\eta(P_{2n+1}^k) = (2n + 1)(2n)^k - \eta(P_{2n+1}^{k-2}), \text{ for all } k, k \geq 3.
\]

**Proof:** i) The proof is similar to that of Proposition 3.1 (ii).

ii) For \( k = 1 \), it is clear that \( \eta(P_{2n+1}^k) = 1 \) by Proposition 2.7 (ii). For \( k = 2 \),
\[
P_{2n+1}^2 = P_{2n+1}^1(P_{2n+1}^1),
\]
is a rooted product of \( P_{2n+1}^1 \) and \( P_{2n+1}^1 \). To prove that \( \eta(P_{2n+1}^2) = 1 \), let \( x_{i,j}, \ i, j = 1, 2, \ldots, 2n + 1 \) be a weighting for the vertex \( v_{i,j} \) in \( P_{2n+1}^2, n = 1, 2, \ldots \), as indicated in Figure 3.3.

![Figure 3.3. A weighting of \( P_{2n+1}^2 \), where the rooted vertex has a non-zero weight.](image)

Then, from the condition that \( \sum_{w \in N_G(v)} f(w) = 0 \), for all \( v \) in \( P_{2n+1}^2, n = 1, 2, \ldots \), we have:

For all \( i, i = 1, 2, \ldots, 2n + 1 \),
\[
x_{i,2n} = 0.
\]  

Because \( x_{i,2n} \) are the neighbors of the end vertices.

Also, for all \( i, j \), for which \( i, j = 1, 2, \ldots, 2n + 1 \)
Nullity and Bounds to the Nullity of Dendrimer Graphs

\[ x_{i,j} = -x_{i,j+2} \quad \text{and} \quad x_{i,j} = -x_{i+2,j} \quad \ldots (3.13) \]

Thus, from Equations (2.13) and (2.14), we get:

For \( i = 1, 2, \ldots, 2n + 1 \) and \( j = 2, 4, \ldots, 2n \).

\[ x_{i,j} = 0. \quad \ldots (3.14) \]

Hence, from the condition that \( \sum_{w \in N_G(v)} f(w) = 0 \), for all \( v \in P_{2n+1}^2 \) and Equations (3.13) and (3.14), we get:

\[ x_{1,2} + x_{2,1} = 0 \quad \Rightarrow \quad x_{2,1} = 0. \]

While, from Equation (3.13) and for all \( i \) and \( j \), for which \( i = 2, 4, \ldots, 2n \) \( j = 1, 2, \ldots, 2n + 1 \), we have: \( x_{i,j} = 0 \).

Therefore, each vertex of \( P_{2n+1}^2 \) has the weight 0 or \( x_{1,2n+1} \) or \( -x_{1,2n+1} \). Thus, any high zero-sum weighting of \( P_{2n+1}^2 \) will use only one non-zero variable, say \( x_{1,2n+1} \). Therefore, \( \eta(P_{2n+1}^2) = 1 \) where the rooted vertex has non-zero weight, and for \( k \geq 3 \), similar steps for the proof hold as in the case where \( k = 2 \). Thus, any high zero-sum weighting of \( P_{2n+1}^2, k \geq 3 \), will use only one non-zero variable. Hence, \( \eta(P_{2n+1}^k) = 1 \).

iii) For \( k = 1 \), it is clear that \( \eta(P_{2n+1}^2) = 1 \) by Proposition 2.7 (ii). For \( k = 2 \), \( P_{2n+1}^2 = P_{2n+1}^2(P_{2n+1}^2) \), is a rooted product of \( P_{2n+1}^2 \) and \( P_{2n+1}^2 \). To prove that \( \eta(P_{2n+1}^2) = 2n + 1 \), where the rooted vertex has zero weight, let the rooted vertex is neighbor of end vertex in \( P_{2n+1}^2 \), and let \( x_{i,j}, i, j = 1, 2, \ldots, 2n + 1 \) be a weighting for \( P_{2n+1}^2, n = 1, 2, \ldots \), as indicated in Figure 3.4.

![Figure 3.4. A weighting of \( P_{2n+1}^2 \) where the rooted vertex has a zero weight.](image)

Then, from the condition that \( \sum_{w \in N_G(v)} f(w) = 0 \), for all \( v \in P_{2n+1}^2, n = 1, 2, \ldots \), we have:

For all \( i, j \), for which \( i = 1, 2, \ldots, 2n + 1 \) and \( j = 2, 4, \ldots, 2n \).

\[ x_{i,j} = 0. \quad \ldots (3.15) \]

And, for all \( i \) and \( j \), for which \( i = 1, 2, \ldots, 2n + 1 \) and \( j = 1, 3, \ldots, 2n + 1 \).

\[ x_{i,j} = -x_{i,j+2}. \quad \ldots (3.16) \]

Therefore, for each \( i \) we use one variable. Thus, the maximum number of non-zero independent variables used in a high zero-sum weighting of \( P_{2n+1}^2 \) is equal to \( 2n + 1 \). Hence, \( \eta(P_{2n+1}^2) = 2n + 1 \).
On the other hand, $P_{2n+1}^2 = P_{2n+1}(P_{2n+1})$, since $P_{2n+1}$ has $2n+1$ vertices to be attachment and each vertex adds (increases) one to the nullity, thus:

$$\eta(P_{2n+1}^2) = (2n+1)*1 = 2n+1.$$

For $k = 3$, use the iteration $P_{2n+1}^k = P_{2n+1}(P_{2n+1})$. Since, $P_{2n+1}^2$ is a dendrimer graph having $2n+1$ paths and each path has $2n$ vertices to be attachment, thus we attach $P_{2n+1}$ to $(2n+1)(2n)$ vertices. But, each copy of $P_{2n+1}$ adds (increases) one to the nullity of a dendrimer graph, and together the variable used in a high zero-sum weighting of $P_{2n+1}$. Therefore,

$$\eta(P_{2n+1}^2) = (2n+1)(2n) + \eta(P_{2n+1})$$

$$= (2n+1)(2n) + 1.$$

Similarly, we have: $\eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + \eta(P_{2n+1}^{k-2})$, for each $k$, $k \geq 3$.

**Corollary 3.4:** For a dendrimer graph $P_{2n+1}^k$, $k \geq 2$, $n=1,2,...$, and the rooted vertex has zero weight, we have:

i) If $k$ is odd, $k \geq 3$, then: $\eta(P_{2n+1}^k) = 1 + (2n+1) \sum_{i=1}^{k-1} (2n)^{2i-1}$.

ii) If $k$ is even, $k \geq 2$, then: $\eta(P_{2n+1}^k) = (2n+1) \sum_{i=0}^{k-2} (2n)^{2i}$.

**Proof:** i) From Proposition 3.3 (iii), we have:

$$\eta(P_{2n+1}) = 1, \quad \eta(P_{2n+1}^2) = 2n+1,$$

and

$$\eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + \eta(P_{2n+1}^{k-2}),$$

for each $k$, $k \geq 3$.

$$\therefore \eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \eta(P_{2n+1}^{k-4})$$

$$= (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \ldots + (2n+1)(2n)^1 + \eta(P_{2n+1})$$

$$= (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \ldots + (2n+1)(2n)^1 + 1$$

$$= (2n+1)^{k-2} + (2n)^{k-4} + \ldots + (2n)^1 + 1$$

$$= (2n+1)^{k-1} + 1.$$ 

$$\therefore \eta(P_{2n+1}^k) = 1 + (2n+1) \sum_{i=1}^{k-1} (2n)^{2i-1},$$

if $k$ is odd, $k \geq 3$.

ii) From Proposition 3.3 (iii), we have:

$$\eta(P_{2n+1}) = 1, \quad \eta(P_{2n+1}^2) = 2n+1,$$

and

$$\eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + \eta(P_{2n+1}^{k-2}),$$

for each $k$, $k \geq 3$.

$$\therefore \eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \eta(P_{2n+1}^{k-4})$$

$$= (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \ldots + (2n+1)(2n)^2 + \eta(P_{2n+1}^2)$$

$$= (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \ldots + (2n+1)(2n)^2 + (2n+1)^1$$

$$= (2n+1)^{k-2} + (2n)^{k-4} + \ldots + (2n)^2 + 1$$

$$= (2n+1)^k + 1.$$ 

$$\therefore \eta(P_{2n+1}^k) = (2n+1) \sum_{i=0}^{k-2} (2n)^{2i},$$

if $k$ is even, $k \geq 2$. 

80
Nullity and Bounds to the Nullity of Dendrimer Graphs

\[(2n+1) \sum_{i=0}^{\frac{k-3}{2}} (2n)^{2i}.\]

\[\therefore \eta(P_{2n+1}^k) = (2n+1) \sum_{i=0}^{\frac{k-3}{2}} (2n)^{2i}, \text{ if } k \text{ is even, } k \geq 2.\]

The nullities of dendrimers of complete graphs are determined in the next proposition.

**Proposition 3.5:** For a dendrimer graph \(K_p^k, k \geq 1\) we have:

i) If \(p = 3\), then \(\eta(K_p^1) = 0\) for all \(k, 2 \neq k \geq 1\). And \(\eta(K_p^2) = 1\).

ii) If \(p \geq 4\), then \(\eta(K_p^k) = 0\) for all \(k, k \geq 1\).

**Proof:** The proof is immediate by Proposition 3.1.

For Complete bipartite graph \(K_{m,n}, m,n \geq 2\) has exactly 3 distinct eigenvalues, while the dendrimer \(K_{m,n}^k, k \geq 2\), loses this property.

**Proposition 3.6:** For a dendrimer graph \(K_{m,n}^k, k \geq 1, m,n \geq 2\), we have:

\[\eta(K_{m,n}^k) = m+n-2,\]

and

\[\eta(K_{m,n}^k) = \eta(K_{m,n}^{k-1}) + (m+n)(m+n-1)^{k-2}(m+n-3), \text{ for all } k, k \geq 2.\]

**Proof:** For \(k=1\), it is clear that \(\eta(K_{m,n}^1) = m+n-2\) by Prop. 2.7(iv). For \(k = 2,\)

\[K_{m,n}^2 = K_{m,n}(K_{m,n}),\]

is a rooted product of \(K_{m,n}\) and \(K_{m,n}\). To prove that

\[\eta(K_{m,n}^2) = (m+n-2) + (m+n)(m+n-3),\]

which is the number of independent variables used in a high zero-sum weighting for \(K_{m,n}^2\). For \(k \geq 3\), we use the iteration

\[K_{m,n}^3 = K_{m,n}(K_{m,n}^2),\]

since \(K_{m,n}^2\) is a dendrimer graph having \((m+n)\) complete bipartite graphs \(K_{m,n}\), and each graph has \((m+n-1)\) vertices to be attached; hence, we attach \(K_{m,n}\) to \((m+n)(m+n-1)\) vertices, but each copy of \(K_{m,n}\) adds (increases) \((m+n-3)\) to the nullity of the dendrimer graph.

Thus, \(\eta(K_{m,n}^3) = \eta(K_{m,n}^2) + (m+n)(m+n-1)^{k-2}(m+n-3), \text{ for all } k, k \geq 2.\)

**Corollary 3.7:** For a dendrimer graph \(K_{m,n}^k, k \geq 2, m,n \geq 2,\)

\[\eta(K_{m,n}^k) = m+n-2,\]

and

\[\eta(K_{m,n}^k) = \eta(K_{m,n}^{k-1}) + (m+n)(m+n-1)^{k-2}(m+n-3), \text{ for all } k, k \geq 2.\]

\[\therefore \eta(K_{m,n}^k) = \eta(K_{m,n}^{k-1}) + (m+n)(m+n-1)^{k-3}(m+n-3)
+ (m+n)(m+n-1)^{k-2}(m+n-3)
+ ... + (m+n)(m+n-1)^{k-3}(m+n-3)
+ (m+n)(m+n-1)^{k-2}(m+n-3).\]
\[\begin{align*}
&= (m + n - 2) + (m + n)(m + n - 3) \\
&\quad + (m + n)(m + n - 1)^{k-1}(m + n - 3) \\
&\quad + \ldots + (m + n)(m + n - 1)^{k-3}(m + n - 3) \\
&\quad + (m + n)(m + n - 1)^{k-2}(m + n - 3) \\
&= (m + n - 2) + (m + n)(m + n - 3)[1 + (m + n - 1)^{k-1} + \ldots + (m + n - 1)^{k-3} + (m + n - 1)^{k-2}] \\
&= (m + n - 2) + (m + n)(m + n - 3)\frac{(m + n - 1)^{k-1} - 1}{m + n - 2}, \text{ for all } k, \ k \geq 2. \blacksquare
\end{align*}\]

**Star graphs** are special cases of complete bipartite graphs, namely \(S_{1,n-1}\) is \(K_{1,n-1}\) with a partite set consisting of a single vertex called the central vertex.

**Proposition 3.8:** For a dendrimer graph \(S_{k,n-1}\), \(k \geq 1, \ n \geq 3\), we have:

i) If the rooted vertex of \(S_{i,n-1}\) is the central vertex, then

\[\eta(S_{i,n-1}) = n - 2, \ \eta(S_{i,n-1}^2) = n(n - 2), \ \text{and} \]

\[\eta(S_{i,n-1}^k) = n(n - 1)^{k-2}(n - 2) + \eta(S_{i,n-1}^{k-2}), \ \text{for all } k, \ k \geq 3.\]

ii) If the rooted vertex of \(S_{i,n-1}\) is a non-central vertex, then

\[\eta(S_{i,n-1}) = n - 2, \ \text{and} \ \eta(S_{i,n-1}^k) = n(n - 1)^{k-2}(n - 3) + \eta(S_{i,n-1}^{k-1}), \ \text{for all } k, \ k \geq 2.\]

**Proof:** i) For \(k = 1\), it is clear that \(\eta(S_{i,n-1}) = n - 2\) by Proposition 2.7 (iv). For \(k = 2\), \(S_{1,n-1}^2 = S_{1,n-1}(S_{1,n-1})\), is a rooted product of \(S_{1,n-1}\) and \(S_{1,n-1}\). To prove that \(\eta(S_{1,n-1}^2) = n(n - 2)\); let \(x_{i,j}, i, j = 1, 2, \ldots, n\) be a weighting for \(S_{1,n-1}^2\), as indicated in Figure 3.5.

![Figure 3.5](https://via.placeholder.com/150)

**Figure 3.5.** A weighting of \(S_{1,n-1}^2\), where the rooted vertex of \(S_{1,n-1}\) is the central vertex.

Then, from the condition that \(\sum_{w \in N_G(v)} f(w) = 0\), for all \(v \in S_{1,n-1}^2\), we have:

\[x_{1,n} = x_{2,n} = \ldots = x_{n,n} = 0\]  \hfill \ldots(3.17)

And,
Nullity and Bounds to the Nullity of Dendrimer Graphs

\[
x_{1,1} + x_{1,2} + \ldots + x_{1,n-1} = 0 \\
x_{2,1} + x_{2,2} + \ldots + x_{2,n-1} = 0 \\
\vdots \\
x_{n,1} + x_{n,2} + \ldots + x_{n,n-1} = 0
\]

Then,
\[
x_{1,n-1} = -x_{1,1} - x_{1,2} - \ldots - x_{1,n-2} \\
x_{2,n-1} = -x_{2,1} - x_{2,2} - \ldots - x_{2,n-2} \\
\vdots
\]

\[
x_{n,n-1} = -x_{n,1} - x_{n,2} - \ldots - x_{n,n-2}
\]

Then, from Equation (3.18), the number of independent variables used in a high zero-sum weighting of \( S^n_{l,n-1} \) is equal to \( n(n-2) \).

Hence, \( \eta(S^n_{l,n-1}) = n(n-2) \).

For \( k = 3 \), use the iteration \( S^3_{l,n-1} = S^2_{l,n-1}(S^1_{l,n-1}) \), since \( S^2_{l,n-1} \) is a dendrimer graph having \( n \) star graphs \( S_{l,n-1} \) and each graph has \( n-1 \) vertices to be attachment, thus we attach \( S_{l,n-1} \) to \( n(n-1) \) vertices. But also, each copy of \( S_{l,n-1} \) adds (increases) \( (n-3) \) to the nullity of a dendrimer graph, together the variable used in a high zero-sum weighting of \( S_{l,n-1} \).

Therefore,
\[
\eta(S^3_{l,n-1}) = n(n-1)(n-2) + \eta(S^1_{l,n-1})
\]

Similarly, we have:
\[
\eta(S^k_{l,n-1}) = n(n-1)^{k-2}(n-2) + \eta(S^{k-2}_{l,n-1}), \text{ for each } k, \ k \geq 3.
\]

(Proposition 3.8 (i), we have:
\[
\eta(S^3_{l,n-1}) = (n-2) + n(n-2) \sum_{i=1}^{k} (n-1)^{2i-1}.
\]

(ii) If \( k \) is even, \( k \geq 2 \), and the rooted vertex of a graph \( H = S_{l,n-1} \) is its central vertex,
then:
\[
\eta(S^k_{l,n-1}) = n(n-2) \sum_{i=0}^{\frac{k-1}{2}} (n-1)^{2i}.
\]

(iii) For all \( k, \ k \geq 2 \), if the rooted vertex of a graph \( H = S_{l,n-1} \) is a non central vertex,
then,
\[
\eta(S^k_{l,n-1}) = (n-2) + n(n-3)^{k-1} - 1, \text{ for all } k, \ k \geq 2.
\]

Proof: i) From Proposition 3.8 (i), we have:
\[
\eta(S_{l,n-1}) = n - 2, \ \eta(S^2_{l,n-1}) = n(n-2), \text{ and}
\]
\[
\eta(S^k_{l,n-1}) = n(n-1)^{k-2}(n-2) + \eta(S^{k-2}_{l,n-1}), \text{ for all } k, \ k \geq 3.
\]

\[
\eta(S^k_{l,n-1}) = n(n-1)^{k-2}(n-2) + n(n-1)^{k-4}(n-2) + \eta(S^{k-4}_{l,n-1})
\]

\[
\vdots
\]

83
\[
= n(n-1)^{k-2} (n-2) + n(n-1)^{k-4} (n-2) + \ldots + n(n-1)(n-2) + \eta(S_{1,n-1})
\]
\[
= n(n-1)^{k-2} (n-2) + n(n-1)^{k-4} (n-2) + \ldots + n(n-1)(n-2) + (n-2)
\]
\[
= n(n-2) \left[ (n-1)^{k-2} + (n-1)^{k-4} + \ldots + (n-1) \right] + (n-2)
\]
\[
= n(n-2) \sum_{i=1}^{k-1} (n-1)^{2i-1} + (n-2) .
\]
\[
\therefore \quad \eta(S_{1,n-1}^k) = (n-2) + (n-2) \sum_{i=1}^{k-1} (n-1)^{2i-1} , \text{ if } k \text{ is odd, } k \geq 3 .
\]

**ii)** From Proposition 3.8 (i), we have:
\[
\eta(S_{2,n-1}^2) = n(n-2) , \text{ and }
\]
\[
\eta(S_{2,n-1}^k) = n(n-1)^{k-2} (n-2) + \eta(S_{2,n-1}^{k-2}) , \text{ for each } k, k \geq 2
\]
\[
\therefore \quad \eta(S_{2,n-1}^k) = n(n-1)^{k-2} (n-2) + n(n-1)^{k-4} (n-2) + \eta(S_{2,n-1}^{k-4})
\]
\[
\therefore \quad \eta(S_{2,n-1}^k) = n(n-1)^{k-2} (n-2) + n(n-1)^{k-4} (n-2) + \ldots + n(n-1)^2 (n-2) + \eta(S_{2,n-1}^2)
\]
\[
= n(n-1)^{k-2} (n-2) + n(n-1)^{k-4} (n-2) + \ldots + n(n-1)^2 (n-2) + (n-2)
\]
\[
= n(n-2) \left[ (n-1)^{k-2} + (n-1)^{k-4} + \ldots + (n-1)^2 + 1 \right]
\]
\[
= n(n-2) \sum_{i=0}^{k-2} (n-1)^{2i} .
\]
\[
\therefore \quad \eta(S_{2,n-1}^k) = n(n-2) \sum_{i=0}^{k-2} (n-1)^{2i} , \text{ if } k \text{ is even, } k \geq 2 .
\]

**iii)** From Proposition 3.8 (ii), we have:
\[
\eta(S_{1,n-1}^1) = n-2 , \text{ and } \eta(S_{1,n-1}^k) = n(n-1)^{k-2} (n-3) + \eta(D^{k-1}) , \text{ for all } k, k \geq 2
\]
\[
\therefore \quad \eta(S_{1,n-1}^k) = n(n-1)^{k-2} (n-3) + n(n-1)^{k-3} (n-3) + \eta(S_{1,n-1}^{k-3})
\]
\[
\therefore \quad \eta(S_{1,n-1}^k) = n(n-1)^{k-2} (n-3) + n(n-1)^{k-3} (n-3) + \ldots + n(n-1)(n-3) + \eta(S_{1,n-1}^1)
\]
\[
= n(n-1)^{k-2} (n-3) + n(n-1)^{k-3} (n-3) + \ldots + n(n-1)(n-3) + (n-2)
\]
\[
= n(n-3) \left[ (n-1)^{k-2} + (n-1)^{k-3} + \ldots + (n-1) \right] + (n-2)
\]
\[
= (n-2) + n(n-3) \frac{(n-1)^{k-1} - 1}{n-2} .
\]
\[
\therefore \quad \eta(S_{1,n-1}^k) = (n-2) + n(n-3) \frac{(n-1)^{k-1} - 1}{n-2} , \text{ for all } k, k \geq 2 .
\]

4. **Upper Bounds for the Nullity of Coalescence Graphs**

In this section, we shall introduce and prove a lower and an upper bound for the nullity of the coalescence graph \( G_1 \circ G_2 \).

**Proposition 4.1:** For any singular graphs \( G_1 \) and \( G_2 \),
\[
\eta(G_1) + \eta(G_2) - 1 \leq \eta(G_1 \circ G_2) \leq \eta(G_1) + \eta(G_2) + 1
\]
**Proof:** Let $G_1$ and $G_2$ be two singular graphs of orders $p_1$ and $p_2$, respectively, thus first we label the vertices of $G_1$ by $u_1, u_2, \ldots, u_{p_1}$, with a high zero- sum weighting $x_1, x_2, \ldots, x_{p_1}$ and the vertices of $G_2$ by $v_1, v_2, \ldots, v_{p_2}$, with a high zero- sum weighting $y_1, y_2, \ldots, y_{p_2}$.

Assume that $u_1$ and $v_1$ are rooted vertices of $G_1$ and $G_2$ respectively. Then equality holds at the left if either or both rooted vertices are non-zero weighted because there exists a high zero- sum weighting for $G_1 \circ G_2$ which is the enlargement of high zero- sum weightings for $G_1$ and $G_2$ reducing or vanishing one non-zero weight at the identification vertex. See Figure 4.1 where $G_1 \circ G_2 = P_3$.

Moreover, strictly holds at the left side if both rooted vertices have zero weights in their high zero- sum weightings, because there exists a zero- sum weighting which is the union of both high zero-sum weightings of $G_1$ and $G_2$.

Equality holds at the right side if both rooted vertices are cut vertices with zero weights in their high zero- sum weightings, and each component obtained with a deleting of a rooted cut vertex is singular, because there exists a high zero- sum weighting for $G_1 \circ G_2$ that uses an extra independent variable further than the variables used in high zero- sum weightings of $G_1$ and $G_2$. See Figure 4.2.

Moreover, strictly holds at the right side if one rooted vertices does not satisfy the condition of equality as indicated above. ■

**Note:** Let $w$ be the identification vertex $w = (u = v)$ of $G = G_1 \circ G_2$. Then, by interlacing Theorem [2, p314], $|\eta(G) - \eta(G - w)| \leq 1$ i.e

$$|\eta(G) - \eta(G - u) - \eta(G - v)| \leq 1 \quad \forall \ u \in G_1, v \in G_2.$$  

Hence,

$$\eta(G_1 - u) + \eta(G_2 - v) - 1 \leq \eta(G_1 \circ G_2) \leq \eta(G_1 - u) + \eta(G_2 - v) + 1.$$
REFERENCES


