Nullity and Bounds to the Nullity of Dendrimer Graphs

Khidir R. Sharaf **Didar A. Ali**
khidirsharaf@yahoo.com *Didar.math@yahoo.com*
Department of Mathematics
faculty of Science, University of Zakho,
Zakho - Kurdistan-Region, Iraq

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ABSTRACT

In this paper, a high zero-sum weighting is applied to evaluate the nullity of a dendrimer graph for some special graphs such as cycles, paths, complete graphs, complete bipartite graphs and star graphs.

Finally, we introduce and prove a sharp lower and a sharp upper bound for the nullity of the coalescence graph of two graphs.

Keywords: Graph spectra, Nullity of graphs

درجة الشذوذ وحدود درجة الشذوذ لبيانات Dendrimer

خدر شرف **ديدار علي**
قسم الرياضيات
كلية العلوم، جامعة زاخو،
زاخو، إقليم كردستان، العراق

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الملخص

في هذا البحث, تم تطبيق تقنية التوزين العالي لإحتساب درجة الشذوذ للبيان(D^k dendrimer) إذ أن D بيان خاص, كالدارة, الدرب, البيان التام, البيان الثنائي التجزئة التام أوالنجمة.أخيراً , وضعنا وأثبتنا قيود حادة دنيا وعليا للبيان $G_1 \circ G_2$.

الكلمات المفتاحية: أطياف البيان، درجة الشذوذ للبيان.

1. Introduction

The characteristic polynomial of the adjacency matrix A(G) is said to be the characteristic polynomial of the graph G, denoted by $\varphi(G;x)$. The eigenvalues of A(G) are said to be the eiegenvalues of the graph G, the occurrence of zero as an eigenvalue in the spectrum of the graph G is called the "nullity" of G denoted by η(G). Brown and others [4] proved that a graph G is singular if, and only if, G possesses a non-trivial zero-sum weighting, and asked, what causes a graph to be singular and what are the effects of this on its properties. Rashid [11] proved that a high zero-sum weighting $M_v(G)$ of a graph G, that is (the maximum number of non zero independent variables used in a high zero- sum weighting for a graph G, is equal to the nullity of G) It is known that $0 \leq \eta(G) \leq p-2$ if G is a non empty graph with p vertices. Cheng and Liu [5] proved that if G has p vertices with no isolated vertices, then η(G) = p-2 if, and only if, G is isomorphic to a complete bipartite graph $K_{m,n}$, and η(G) = p - 3 if, and only if, G is isomorphic to a complete 3 partite graph $K_{a,b,c}$. Omidi [10] found some lower bounds for the nullity of graphs and proved that among bipartite graphs with p vertices, q edges and maximum degree Δ which do not have any cycle of length a multiple of 4 as a subgraph, the greatest nullity is $p - 2\lceil q/\Delta \rceil$.

In this paper, we continue the research along the same lines. We derive formulas to determine the nullity of dendrimer graphs.

2 **Definition and Preliminary Results**

Definition 2.1: [5, p.16] and [8] A vertex weighting of a graph G is a function f: V(G) →R where R is the set of real numbers, which assigns a real number (weight) to each vertex. The weighting of G is said to be non-trivial if there is, at least, one vertex v∈V(G) for which f(v) ≠ 0.

Definition 2.2: [5, p.16] A non-trivial vertex weighting of a graph G is called a zero-sum weighting provided that for each v∈V(G), ∑f(w) = 0, where the summation is taken over all w∈NG(v).

Clearly, the following weighting for G is a non-trivial zero-sum weighting where x1, x2, x3, x4, and x5 are weights and provided that (x1, x2, x3, x4, x5) ≠ (0, 0, 0, 0, 0) as indicated in Figure 2.1.

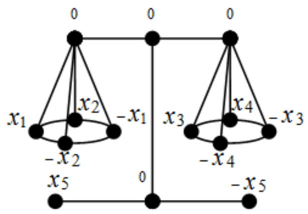


Figure 2.1. A non-trivial zero-sum weighting for a graph G.

theorem 2. 3: [4] a graph g is singular if, and only if, there is a non-trivial zero-sum weighting for g.■

Hence, the graph G depicted in Figure 2.1 is singular.Out of all zero-sum weightings of a graph G, a high zero-sum weighting of G is one that uses maximum number of non-zero independent variables.

proposition 2.4: [6, p.35] and [8] in any graph g, the maximum number mv(g) of non-zero independent variables in a high zero-sum weighting equals the number of zeros as an eigenvalues of the adjacency matrix of g, (i.e. mv(g) = η(g)).■

In Figure 2.1, the weighting for the graph G is a high zero-sum weighting that uses 5 independent variables, hence, η(G) = 5.

The complement of the disjoint union of m edges is called a cocktail graph and is denoted by CP(m) = (mK2)c = K2,2,…,2 = Km(2).

Proposition 2. 5: [6, p.20] The spectrum of the cocktail graph CP(m) is:

$$S_p(CP(m)) = \begin{pmatrix} 2m-2 & 0 & -2 \\ 1 & m & m-1 \end{pmatrix} \text{,thus } \eta(CP(m)) = \begin{cases} 2, & if\ m=1, \\ m, & if\ m>1. \end{cases}$$

Proposition 2.6: [2] The adjacency matrix of the wheel graph Wp, A(Wp), has eigenvalues 1+$\sqrt{p}$, 1-$\sqrt{p}$ and 2cos$\frac{2\pi r}{p-1}$, r = 0, 1, …, p-2. Hence,η(Wp)= 2 if p=1(mod4) and o otherwise.

Proposition 2.7: [4, p.72]i) The eigenvalues of the cycle Cp are of the form 2cos$\frac{2\pi r}{p}$, r = 0, 1, …, p-1. According to this, η(CP)= 2 if p=0(mod4) and 0 otherwise.

ii) The eigenvalues of the path Pp are of the form 2cos$\frac{\pi r}{p+1}$, r =1,2, … p. And thus, η(PP)=1 if p is odd and 0 otherwise.

iii) The spectrum of the complete graph Kp, consists of p-1 and -1 with multiplicity p-1.

iv) The spectrum of the complete bipartite graph Km,n , consists of √mn, -√mn and zero m+n-2 times

Corollary 2.8: [4, p.234] If G is a bipartite graph with an end vertex, and if H is an induced subgraph of G obtained by deleting this vertex together with the vertex adjacent to it, then η(G) = η(H). ■

Corollary 2.9: [4, p.235] Let G1 and G2 be two bipartite graphs in which η(G1) = 0. If the graph G is obtained by joining an arbitrary vertex of G1 by an edge to an arbitrary vertex of G2, then η(G) = η(G2).■

Coalescence Graphs

To **identify** nonadjacent vertices u and v of a graph G is to replace the two vertices by a single vertex incident to all the edges which are incident in G to either u or v. Denote the resulting graph by G/{u, v}. To **contract** an edge e of a graph G is to delete the edge and then (if the edge is a link) identify its ends. The resulting graph is denoted by G/e.

Definition 2.10: [7] Let (G_1, u) and (G_2, v) be two graphs rooted at vertices u and v, respectively. We attach G_1 to G_2 (or G_2 to G_1) by identifying the vertex u of G_1 with the vertex v of G_2. Vertices u and v are called **vertices of attachment**. The vertex formed by their identification is called the **coalescence vertex**. The resulting graph $G_1 \circ G_2$ is called the **coalescence (vertex identification)** of G_1 and G_2.

Definition 2.11: [7] Let {(G_1, v_1), (G_2, v_2), …, (G_t, v_t)} be a family of not necessary distinct connected graphs with roots v_1, v_2, …, v_t, respectively. A connected graph G= $G_1 \circ G_2 \circ \ldots \circ G_t$ is called the **multiple coalescence** of G_1, G_2,…,G_t provided that the vertices v_1, v_2, …, v_t are identified to reform the coalescence vertex v. The **t-tupple coalescence graph** is denoted by $\overset{|t|}{G}$ is the multiple coalescence of t isomorphic copies of a graph G. In the same ways $G_1 \circ \overset{|t|}{G_2}$ is the multiple coalescence of G_1 and t copies of G_2.

Remark 2.12: [7] All coalescened graphs have v as a common cut vertex. Some graphs and their operations will, herein, be illustrated in Figure 2.2.

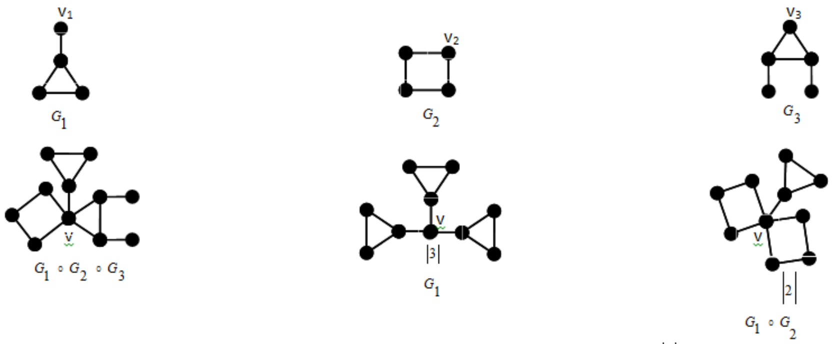


Figure 2.2. Multiple coalescence $G_1 \circ G_2 \circ G_3$, t-tupple coalescence $\overset{|3|}{G_1}$ and coalescence of both $G_1 \circ \overset{|2|}{G_2}$.

Definition 2.13: [7] Let G be a graph consisting of n vertices and L = {H_1, H_2, …, H_n} be a family of rooted graphs. Then, the graph formed by attaching H_k to the k-th ($1 \leq k \leq n$) vertex of G is called the **generalized rooted product** and is denoted by G(L); G itself is called the **core** of G(L). If each member of L is isomorphic to the rooted graph H, then the graph G(L) is denoted by G(H). Recall G_1, G_2 and G_3 from Figure 2.2. Then, we have

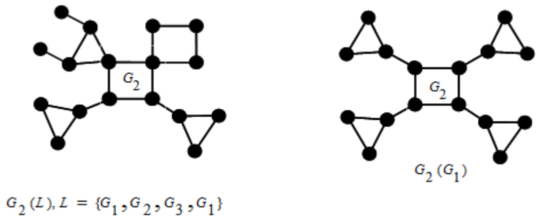


Figure 2.3. Generalized rooted product graphs

Definition 2.14: [7] The generalization of the rooted product graphs is called the **F-graphs**, which are consecutively iterated rooted products defined as: $\overset{0}{F} = K_1$, $\overset{1}{F} = G =$ H, $\overset{2}{F} =$ G(H), …, $\overset{s+1}{F} = \overset{s}{F}$ (H), $s \geq 1$.

Definition 2.15: [7] A family of **dendrimers** $\overset{k}{D}$ ($k \geq 0$) is just a rooted product graph which is defined as follows:

$\overset{0}{D} = K_1$, $\overset{1}{D} = G = H$, $\overset{2}{D}$ is the rooted product of G and H, in which some attachments of H are not made (i.e., H attached to the vertices of G which are not attached before). In general, $\overset{k+1}{D}$ $(k \geq 1)$ is constructed from $\overset{k}{D}$, and the number of copies of H attached to $\overset{k}{D}$ obeys some fixed generation law. Hence, $\overset{k+1}{D}$ is $\overset{k}{D}$ with G attached to each vertex of $\overset{k}{D}$ which is not in $\overset{k-1}{D}$, that is to each $u \in V(\overset{k}{D}) - V(\overset{k-1}{D})$.), $k \geq 1$.

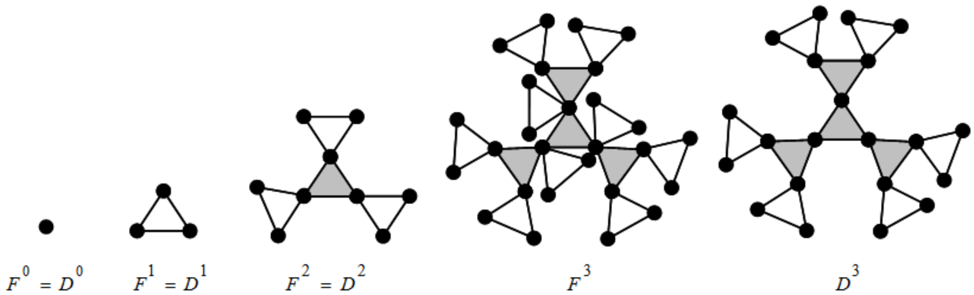


Figure 2.4. F-graphs and Dendrimer graphs $\overset{k}{D}$, where G = H = C_3.

3 Nullity of Dendrimer Graphs

In this section, we determine the nullity of dendrimer graphs D^k, $k \geq 0$, where $D^1 = G$ of some known graphs such as C_p, P_p, K_p and $K_{m,n}$. In each case, we consider that the nullity of the dendrimer graph D^0 is defined to be, $\eta(D^0) = \eta(K_1) = 1$. The dendrimer C_p^k for the **cycle** C_p is a connected graph with order

$p(C_p^k) = p + p(p-1) + p(p-1)^2 + ... + p(p-1)^{k-1} = \sum_{i=1}^{k} P(p-1)^{i-1}$. And size

$$q(C_p^k) = q + pq + p(p-1)q + ... + p(p-1)^{k-2} q$$

$$= p + p^2 + p^2(p-1) + ... + p^2(p-1)^{k-2} = p + p^2 \sum_{t=2}^{k} (p-1)^{t-2}.$$

Moreover, the diameter of C_p^k is $(2k-1).diam(C_p)$. Also for $k > 1$, the degrees of each vertex of C_p^k is either 2 or 4.

Proposition 3.1: For a dendrimer graph C_p^k, $k \geq 1$, we have:

i) If $p = 4n$, $n = 1, 2, \ldots$, then $\eta(C_{4n}^1) = 2$.

And for all k, $k \geq 2$, $\eta(C_{4n}^k) = \eta(C_{4n}^{k-1}) + 4n(4n-1)^{k-2}$.

ii) If $p = 4n+2$, $n = 1, 2, \ldots$, then $\eta(C_{4n+2}^k) = 0$, for all k, $k \geq 1$.

iii) If $p = 4n-1$, $n = 1, 2, \ldots$, then $\eta(C_{4n-1}^1) = 0$, $\eta(C_{4n-1}^2) = 1$.

And for all k, $k \geq 3$, $\eta(C_{4n-1}^k) = 0$.

iv) If $p = 4n+1$, $n = 1, 2, \ldots$, then $\eta(C_{4n+1}^k) = 0$. for all k, $k \geq 1$.

Proof: i) For $k = 1$ it is clear that $\eta(C_{4n}) = 2$, $n = 1, 2, \ldots$, by Proposition 2.7 (i). For $k = 2$, $C_{4n}(C_{4n})$, is a rooted product of C_{4n} and C_{4n}. So we need to prove that $\eta(C_{4n}^2) = 2 + 4n$. Let $x_{i,j}$, $i, j = 1, 2, \ldots, 4n$ be a weighting for the vertex $v_{i,j}$ in C_{4n}^2, $n = 1, 2, \ldots$, as indicated in Figure 3.1

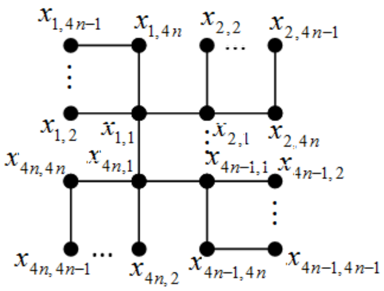


Figure 3.1. A weighting of C_{4n}^2, $n = 1, 2, \ldots$

From the condition that $\sum_{w \in N_G(v)} f(w) = 0$, for all v in C_{4n}^2, $n = 1, 2, \ldots$, we have,{for the cycles identified with the vertices $v_{i,1}$ }.

For $j = 1, 3, \ldots, 4n-3$.

$$\left.\begin{array}{lll} x_{1,j} + x_{1,j+2} = 0 & \Rightarrow & x_{1,j} = -x_{1,j+2} \\ x_{2,j} + x_{2,j+2} = 0 & \Rightarrow & x_{2,j} = -x_{2,j+2} \\ \vdots & & \\ x_{4n,j} + x_{4n,j+2} = 0 & \Rightarrow & x_{4n,j} = -x_{4n,j+2} \end{array}\right\} \quad \ldots(3.1)$$

And, for $j = 2, 4, \ldots, 4n-2$.

$$\left.\begin{array}{lll} x_{1,j} + x_{1,j+2} = 0 & \Rightarrow & x_{1,j} = -x_{1,j+2} \\ x_{2,j} + x_{2,j+2} = 0 & \Rightarrow & x_{2,j} = -x_{2,j+2} \\ \vdots & & \\ x_{4n,j} + x_{4n,j+2} = 0 & \Rightarrow & x_{4n,j} = -x_{4n,j+2} \end{array}\right\} \quad \ldots(3.2)$$

Also, from the condition that $\sum_{w \in N_G(v)} f(w) = 0$, for all v in the central cycle C_{4n}, we have,

For $i = 1, 3, \ldots, 4n-3$.

$$x_{i,1} + x_{i+2,1} = 0 \Rightarrow x_{i,1} = -x_{i+2,1} \quad \ldots(3.3)$$

And, for $i = 2, 4, \ldots, 4n-2$.

$$x_{i,1} + x_{i+2,1} = 0 \Rightarrow x_{i,1} = -x_{i+2,1} \quad \ldots(3.4)$$

Therefore, for each i in the Equations (3.1), (3.2) and (3.4) we have used exactly two non-zero independent variables, one of which in the weight of $x_{i,1}$, where i is odd and the other in the weight of $x_{i,1}$, where i is even. And from Equation (3.2) we have used $4n$ non-zero independent variables.

Thus, the maximum number of non-zero independent variables used in a high zero-sum weighting of $C_{4n}^2, n=1,2,\ldots$, is equal to $2+4n$.

On the other hand, we have $\eta(C_{4n})=2$, $n=1,2,\ldots$, by Lemma 2.7 (i). But $C_{4n}^2=C_{4n}(C_{4n})$, so each identification of a copy of C_{4n} with a vertex of C_{4n} adds (increases) one to the nullity of a dendrimer graph. Since C_{4n} has $4n$ vertices; thus, $4n$ copies of a cycle C_{4n} are identified to C_{4n}.
Therefore, $\eta(C_{4n}(C_{4n}))=\eta(C_{4n})+4n=2+4n$.

For $k\geq 3$, we use the iteration $C_{4n}^2(C_{4n})$. This graph is a rooted product of C_{4n}^2 and C_{4n}. Since C_{4n}^2 is a dendrimer graph having $4n$ cycles and each cycle has $4n-1$ vertices to be identified with new vertices, hence we attach a copy of C_{4n} to $4n(4n-1)$ vertices. Also, each copy of C_{4n} adds (increases) one to the nullity of a dendrimer graph. Therefore, $\eta(C_{4n}^2(C_{4n}))=\eta(C_{4n}^2)+4n(4n-1)=2+4n+4n(4n-1)$.

Similarly, we have, $\eta(C_{4n}^{k-1}(C_{4n}))=\eta(C_{4n}^{k-1})+4n(4n-1)^{k-2}$ where k≥3.

ii) For each k, $k\geq 1$, there exists no non-trivial zero-sum weighting for C_{4n+2}^k, $n=1,2,\ldots$. Thus, by Theorem 2.3, C_{4n+2}^k is non-singular.

iii) For $k=1$, there exists no non-trivial zero-sum weighting for C_{4n-1}, $n=1,2,\ldots$. Thus, by Theorem 2.3, C_{4n-1} is non-singular. For $k=2$, $C_{4n-1}(C_{4n-1})$, is a rooted product of C_{4n-1} and C_{4n-1}. To prove that $\eta(C_{4n-1}(C_{4n-1}))=1$. Let $x_{i,j}$, $i,j=1,2,\ldots,4n-1$ be a weighting for

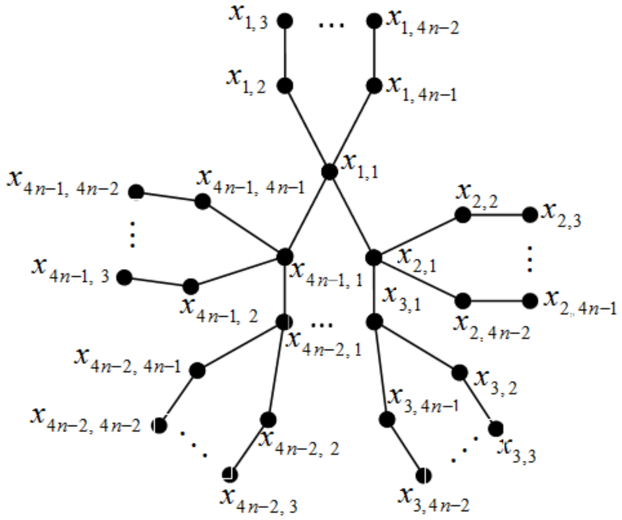


Figure 3.2. A weighting of C_{4n-1}^2, $n=1,2,\ldots$

vertex $v_{i,j}$ in C_{4n-1}^2, $n=1,2,\ldots$, as indicated in Figure 3.2.

Then, from the condition that $\sum_{w\in N_G(v)} f(w)=0$, for all v in C^2_{4n-1}, $n=1,2,\dots$, we have:

For $i=1,2,\dots,4n-1$, and $j=1,3,\dots,4n-3$.

$$x_{i,j}+x_{i,j+2}=0 \quad \Rightarrow \quad x_{i,j}=-x_{i,j+2} \qquad \dots(3.5)$$

And, For $i=1,2,\dots,4n-1$, and $j=2,4,\dots,4n-2$.

$$x_{i,j}+x_{i,j+2}=0 \quad \Rightarrow \quad x_{i,j}=-x_{i,j+2} \qquad \dots(3.6)$$

Hence, from Equations (3.5) and (3.6), we get:

$$x_{1,1}=x_{1,4}=x_{1,5}=x_{1,8}=x_{1,9}=\dots=x_{1,4n-4}=x_{1,4n-3}=-x_{1,2} \qquad \dots(3.7)$$

And $x_{1,2}=x_{1,3}=x_{1,6}=x_{1,7}=\dots=x_{1,4n-2}=x_{1,4n-1}=-x_{1,1}$ $\qquad \dots(3.8)$

Also, from the condition that $\sum_{w\in N_G(v)} f(w)=0$, for all v in C^2_{4n-1}, $n=1,2,\dots$, we have:

$$x_{1,2}+x_{1,4n-1}+x_{2,1}+x_{4n-1,1}=0$$

Since $x_{1,2}=x_{1,4n-1}=-x_{1,1}$, therefore, $x_{2,1}=x_{4n-1,1}=x_{1,1}$ $\qquad \dots(3.9)$

Hence, from Equations (3.7), (3.8) and (3.9), we get

For $i=1,2,\dots,4n-1$.

$$x_{i,1}=x_{i,4}=x_{i,5}=x_{i,8}=x_{i,9}=\dots=x_{i,4n-4}=x_{i,4n-3}=x_{1,1} \qquad \dots(3.10)$$

And, For $i=1,2,\dots,4n-1$.

$$x_{i,2}=x_{i,3}=x_{i,6}=x_{i,7}=\dots=x_{i,4n-2}=x_{i,4n-1}=-x_{1,1} \qquad \dots(3.11)$$

Therefore, each vertex of C^2_{4n-1}, $n=1,2,\dots$ has a weight $x_{1,1}$ or $-x_{1,1}$.

This means that there exists a non-trivial zero-sum weighting for C^2_{4n-1} used exactly one non-zero independent variable in a high zero-sum weighting of C^2_{4n-1}. Hence, $\eta(C^2_{4n-1})=1$.

Finally, the proof of $\eta(C^k_{4n-1})=0$, for $k\geq 3$, is similar to that for k=2.

iv) The proof is similar to that of part (ii).■

Corollary 3.2: For a dendrimer graph C^k_{4n}, $k\geq 2$, $n=1,2,\dots$, we have

$$\eta(C^k_{4n})=2+4n[\sum_{i=0}^{k-2}(4n-1)^i] .$$

Proof: From Proposition 3.1 (i), we have:

$$\eta(C^k_{4n})=\eta(C^{k-1}_{4n})+4n(4n-1)^{k-2} \text{ , for } k\geq 2$$

$$\begin{aligned}
\therefore\ \eta(C^k_{4n}) &= \eta(C^{k-2}_{4n})+4n(4n-1)^{k-3}+4n(4n-1)^{k-2}\\
&= \eta(C^{k-3}_{4n})+4n(4n-1)^{k-4}+4n(4n-1)^{k-3}+4n(4n-1)^{k-2}\\
&\vdots\\
&= \eta(C^2_{4n})+4n(4n-1)^1+\dots+4n(4n-1)^{k-3}+4n(4n-1)^{k-2}\\
&= 2+4n+4n(4n-1)+\dots+4n(4n-1)^{k-3}+4n(4n-1)^{k-2}\\
&= 2+4n[1+(4n-1)+\dots+(4n-1)^{k-3}+(4n-1)^{k-2}]\\
&= 2+4n[\sum_{i=0}^{k-2}(4n-1)^i] .
\end{aligned}$$

$$\therefore \quad \eta(C^k_{4n})=2+4n[\sum_{i=0}^{k-2}(4n-1)^i] \text{ , for } k\geq 2 .■$$

Let P_p be a **path** with usually labeled vertices $v_1, v_2, ..., v_p$. If p is odd, this graph has a non-trivial zero-sum weighting, say $x, 0, -x, 0, ...$, which provides that an odd path is singular. Moreover, the dendrimer P_p^k has order

$p(P_p^k) = p + p(p-1) + p(p-1)^2 + ... + p(p-1)^{k-1}$ and size $q(P_p^k) = p(P_p^k) - 1$.

While, the diameter of D^k depends on the choice of the rooted vertex. Also, the maximum degree will be either 3 or 4 for $k \geq 2$, while the minimum degree is 1 .

In general, $diam(P_p^k) \leq (2k-1)(p-1)$, equality holds if k=1 or the rooted vertex is an end vertex of the path.

Proposition 3.3: For a dendrimer graph P_p^k, $k \geq 1$ we have:

i) If $p = 2n$, $n = 1, 2, ...$, then $\eta(P_{2n}^k) = 0$ for all k, $k \geq 1$.

ii) If $p = 2n+1$, $n = 1, 2, ...$, and the rooted vertex has a non-zero weight , then

$\eta(P_{2n+1}^k) = 1$ for all k, $k \geq 1$.

iii) If $p = 2n+1$, $n = 1, 2, ...$, and the rooted vertex has a zero weight , then

$\eta(P_{2n+1}) = 1$, $\eta(P_{2n+1}^2) = 2n+1$, and

$\eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + \eta(P_{2n+1}^{k-2})$, for all k, $k \geq 3$.

Proof: i) The proof is similar to that of Proposition 3.1 (ii).

ii) For $k = 1$, it is clear that $\eta(P_{2n+1}) = 1$ by Proposition 2.7 (ii). For $k = 2$, $P_{2n+1}^2 = P_{2n+1}(P_{2n+1})$, is a rooted product of P_{2n+1} and P_{2n+1}. To prove that $\eta(P_{2n+1}^2) = 1$, let $x_{i,j}$, $i, j = 1, 2, ..., 2n+1$ be a weighting for the vertex $v_{i,j}$ in P_{2n+1}^2, $n = 1, 2, ...$, as indicated in Figure 3.3.

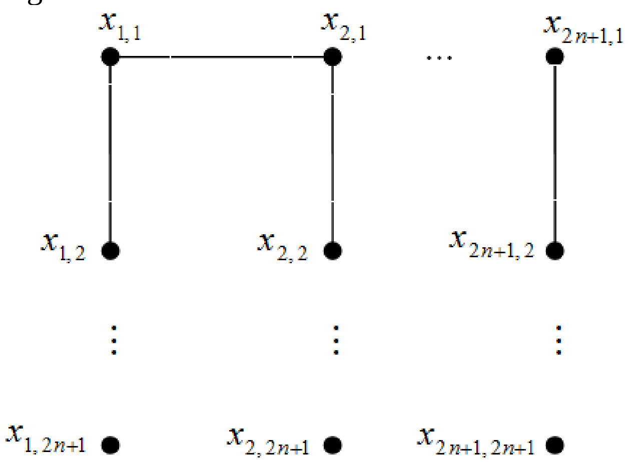


Figure 3.3. A weighting of P_{2n+1}^2 where the rooted vertex has a non-zero weight.

Then, from the condition that $\sum_{w \in N_G(v)} f(w) = 0$, for all v in P_{2n+1}^2, $n = 1, 2, ...$, we have:

For all i, $i = 1, 2, ..., 2n+1$.

$x_{i,2n} = 0$. … (3.12)

Because $x_{i,2n}$ are the neighbors of the end vertices.

Also, for all i, j, for which $i, j = 1, 2, ..., 2n+1$

$x_{i,j} = -x_{i,j+2}$ and $x_{i,j} = -x_{i+2,j}$ …(3.13)

Thus, from Equations (2.13) and (2.14), we get:
For $i = 1, 2, ..., 2n+1$ and $j = 2, 4, ..., 2n$.

$x_{i,j} = 0$. …(3.14)

Hence, from the condition that $\sum_{w \in N_G(v)} f(w) = 0$, for all $v \in P^2_{2n+1}$ and Equations (3.13) and (3.14), we get: $x_{1,2} + x_{2,1} = 0 \Rightarrow x_{2,1} = 0$ .While, from Equation (3.13) and for all I and j, for which $i = 2, 4, ..., 2n$ $j = 1, 2, ..., 2n+1$, we have: $x_{i,j} = 0$ .Therefore, each vertex of P^2_{2n+1} has the weight 0 or $x_{1,2n+1}$ or $-x_{1,2n+1}$ .Thus, any high zero-sum weighting of P^2_{2n+1} will use only one non-zero variable, say $x_{1,2n+1}$.Therefore, $\eta(P^2_{2n+1}) = 1$ where the rooted vertex has non-zero weight, and for $k \geq 3$, similar steps for the proof hold as in the case where $k = 2$.Thus, any high zero-sum weighting of P^k_{2n+1}, $k \geq 3$, will use only one non-zero variable. Hence, $\eta(P^k_{2n+1}) = 1$.

iii) For $k = 1$, it is clear that $\eta(P_{2n+1}) = 1$ by Proposition 2.7 (ii). For $k = 2$, $P^2_{2n+1} = P_{2n+1}(P_{2n+1})$, is a rooted product of P_{2n+1} and P_{2n+1}. To prove that $\eta(P^2_{2n+1}) = 2n+1$, where the rooted vertex has zero weight, let the rooted vertex is neighbor of end vertex in P_{2n+1}, and let $x_{i,j}$, $i, j = 1, 2, ..., 2n+1$ be a weighting for P^2_{2n+1}, $n = 1, 2, ...$, as indicated in Figure 3.4.

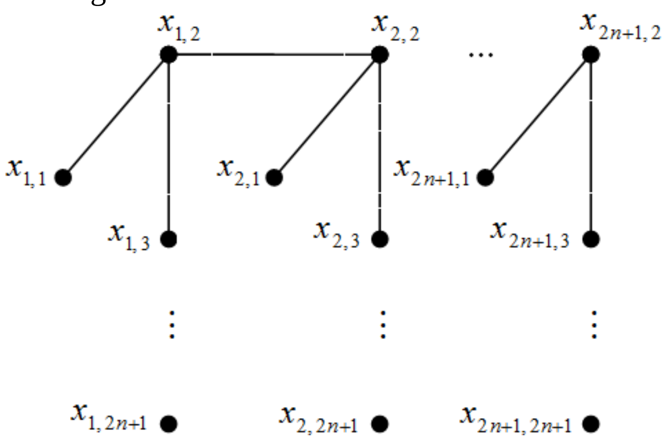


Figure 3.4. A weighting of P^2_{2n+1} where the rooted vertex has a zero weight.

Then, from the condition that $\sum_{w \in N_G(v)} f(w) = 0$, for all v in P^2_{2n+1}, $n = 1, 2, ...$, we have:
For all i, j, for which $i = 1, 2, ..., 2n+1$ and $j = 2, 4, ..., 2n$.

$x_{i,j} = 0$ …(3.15)

And, for all i and j, for which $i = 1, 2, ..., 2n+1$ and $j = 1, 3, ..., 2n+1$.

$x_{i,j} = -x_{i,j+2}$ …(3.16)

Therefore, for each i we use one variable. Thus, the maximum number of non-zero independent variables used in a high zero-sum weighting of P^2_{2n+1} is equal to $2n+1$.
Hence, $\eta(P^2_{2n+1}) = 2n+1$.

On the other hand, $P_{2n+1}^2 = P_{2n+1}(P_{2n+1})$, since P_{2n+1} has $2n+1$ vertices to be attachment and each vertex adds (increases) one to the nullity, thus:

$\eta(P_{2n+1}^2) = (2n+1)*1 = 2n+1$.

For $k=3$, use the iteration $P_{2n+1}^3 = P_{2n+1}^2(P_{2n+1})$. Since, P_{2n+1}^2 is a dendrimer graph having $2n+1$ paths and each path has $2n$ vertices to be attachment, thus we attach P_{2n+1} to $(2n+1)(2n)$ vertices. But, each copy of P_{2n+1} adds (increases) one to the nullity of a dendrimer graph, and together the variable used in a high zero-sum weighting of P_{2n+1}. Therefore,

$$\eta(P_{2n+1}^3) = (2n+1)(2n) + \eta(P_{2n+1})$$
$$= (2n+1)(2n)+1.$$

Similarly, we have: $\eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + \eta(P_{2n+1}^{k-2})$, for each k, $k \ge 3$.■

Corollary 3.4: For a dendrimer graph $P_{2n+1}^k, k \ge 2$, $n = 1,2,...,$ and the rooted vertex has zero weight, we have:

i) If k is odd, $k \ge 3$, then: $\eta(P_{2n+1}^k) = 1 + (2n+1)\sum_{i=1}^{\frac{k-1}{2}} (2n)^{2i-1}$.

ii) If k is even, $k \ge 2$, then: $\eta(P_{2n+1}^k) = (2n+1)\sum_{i=0}^{\frac{k-2}{2}} (2n)^{2i}$.

Proof: i) From Proposition 3.3 (iii), we have:

$\eta(P_{2n+1}) = 1$, $\eta(P_{2n+1}^2) = 2n+1$, and

$\eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + \eta(P_{2n+1}^{k-2})$, for each k, $k \ge 3$

$$\therefore \quad \eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \eta(P_{2n+1}^{k-4})$$
$$\vdots$$
$$= (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \ldots + (2n+1)(2n)^1 + \eta(P_{2n+1})$$
$$= (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \ldots + (2n+1)(2n)^1 + 1$$
$$= (2n+1)[\,(2n)^{k-2} + (2n)^{k-4} + \ldots + (2n)^1\,] + 1$$
$$= (2n+1)\sum_{i=1}^{\frac{k-1}{2}} (2n)^{2i-1} + 1.$$
$$\therefore \quad \eta(P_{2n+1}^k) = 1 + (2n+1)\sum_{i=1}^{\frac{k-1}{2}} (2n)^{2i-1}, \text{ if k is odd, } k \ge 3.$$

ii) From Proposition 3.3 (iii), we have:

$\eta(P_{2n+1}) = 1$, $\eta(P_{2n+1}^2) = 2n+1$, and

$\eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + \eta(P_{2n+1}^{k-2})$, for each k, $k \ge 3$

$$\therefore \quad \eta(P_{2n+1}^k) = (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \eta(P_{2n+1}^{k-4})$$
$$\vdots$$
$$= (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \ldots + (2n+1)(2n)^2 + \eta(P_{2n+1}^2)$$
$$= (2n+1)(2n)^{k-2} + (2n+1)(2n)^{k-4} + \ldots + (2n+1)(2n)^2 + (2n+1)$$
$$= (2n+1)[\,(2n)^{k-2} + (2n)^{k-4} + \ldots + (2n)^2 + 1]$$
$$= (2n+1)[(2n)^0 + (2n)^2 + \ldots + (2n)^{k-4} + (2n)^{k-2}\,]$$

$$= (2n+1)\sum_{i=0}^{\frac{k-2}{2}} (2n)^{2i}\ .$$

$\therefore \quad \eta(P_{2n+1}^{k}) = (2n+1)\sum_{i=0}^{\frac{k-2}{2}} (2n)^{2i}$, if k is even, $k \geq 2$.■

The nullities of dendrimers of complete graphs are determined in the next proposition.

Proposition 3.5: For a dendrimer graph K_p^k, $k \geq 1$ we have:

i) If $p=3$, then $\eta(K_3^k)=0$ for all k, $2 \neq k \geq 1$. And $\eta(K_3^2)=1$.

ii) If $p \geq 4$, then $\eta(K_p^k)=0$ for all k, $k \geq 1$.

Proof: The proof is immediate by Proposition 3.1.■

Every Complete bipartite graph $K_{m,n}$, $m,n,\geq 2$ has exactly 3 distinct eigenvalues, while the dendrimer $K_{m,n}^k, k \geq 2$, loses this property.

Proposition 3.6: For a dendrimer graph $K_{m,n}^k$, $k \geq 1$, $m,n \geq 2$, we have:

$\eta(K_{m,n}) = m+n-2$, and

$\eta(K_{m,n}^k) = \eta(K_{m,n}^{k-1}) + (m+n)(m+n-1)^{k-2}(m+n-3)$, for all k, $k \geq 2$.

Proof: For $k=1$, it is clear that $\eta(K_{m,n}) = m+n-2$ by Prop. 2.7(iv). For $k=2$, $K_{m,n}^2 = K_{m,n}(K_{m,n})$, is a rooted product of $K_{m,n}$ and $K_{m,n}$. To prove that $\eta(K_{m,n}^2) = (m+n-2)+(m+n)(m+n-3)$, which is the number of independent variables used in a high zero-sum weighting for $K_{m,n}^2$. For $k \geq 3$, we use the iteration $K_{m,n}^3 = K_{m,n}^2(K_{m,n})$, since $K_{m,n}^2$ is a dendrimer graph having $(m+n)$ complete bipartite graphs $K_{m,n}$, and each graph has $(m+n-1)$ vertices to be attached; hence, we attach $K_{m,n}$ to $(m+n)(m+n-1)$ vertices, but each copy of $K_{m,n}$ adds (increases) $(m+n-3)$ to the nullity of the dendrimer graph.

Thus, $\eta(K_{m,n}^k) = \eta(K_{m,n}^{k-1}) + (m+n)(m+n-1)^{k-2}(m+n-3)$, for all k, $k \geq 2$.■

Corollary 3.7: For a dendrimer graph $K_{m,n}^k, k \geq 2$, $m,n,\geq 2$, :

$$\eta(K_{m,n}^k) = (m+n-2) + (m+n)(m+n-3)\frac{(m+n-1)^{k-1}-1}{m+n-2}.$$

Proof: From Proposition 2.15, we have: $\eta(K_{m,n}) = m+n-2$, and

$\eta(K_{m,n}^k) = \eta(K_{m,n}^{k-1}) + (m+n)(m+n-1)^{k-2}(m+n-3)$, for all k, $k \geq 2$.

$$\begin{aligned}
\therefore\ \eta(K_{m,n}^k) &= \eta(K_{m,n}^{k-2}) + (m+n)(m+n-1)^{k-3}(m+n-3) \\
&\quad + (m+n)(m+n-1)^{k-2}(m+n-3) \\
&\quad \vdots \\
&= \eta(K_{m,n}^2) + (m+n)(m+n-1)^1(m+n-3) \\
&\quad + \ldots + (m+n)(m+n-1)^{k-3}(m+n-3) \\
&\quad + (m+n)(m+n-1)^{k-2}(m+n-3)
\end{aligned}$$

$$= (m+n-2)+(m+n)(m+n-3)$$
$$+(m+n)(m+n-1)^1(m+n-3)$$
$$+\ldots+(m+n)(m+n-1)^{k-3}(m+n-3)$$
$$+(m+n)(m+n-1)^{k-2}(m+n-3)$$
$$=(m+n-2)+(m+n)(m+n-3)[1+(m+n-1)^1+\ldots+(m+n-1)^{k-3}+(m+n-1)^{k-2}]$$
$$=(m+n-2)+(m+n)(m+n-3)\,\frac{(m+n-1)^{k-1}-1}{m+n-2}$$, for all k, $k \geq 2$.■

Star graphs are special cases of complete bipartite graphs, namely $S_{1,n-1}$ is $K_{1,n-1}$ with a partite set consisting of a single vertex called the central vertex.

Proposition 3.8: For a dendrimer graph $S^k_{1,n-1}$, $k \geq 1$, $n \geq 3$, we have:

i) If the rooted vertex of $S_{1,n-1}$ is the central vertex, then

$\eta(S_{1,n-1}) = n-2$, $\eta(S^2_{1,n-1}) = n(n-2)$, and

$\eta(S^k_{1,n-1}) = n(n-1)^{k-2}(n-2)+\eta(S^{k-2}_{1,n-1})$, for all k, $k \geq 3$.

ii) If the rooted vertex of $S_{1,n-1}$ is a non-central vertex, then

$\eta(S_{1,n-1}) = n-2$,and $\eta(S^k_{1,n-1}) = n(n-1)^{k-2}(n-3)+\eta(S^{k-1}_{1,n-1})$, for all k, $k \geq 2$.

Proof: i) For $k=1$, it is clear that $\eta(S_{1,n-1}) = n-2$ by Proposition 2.7 (iv). For $k=2$, $S^2_{1,n-1} = S_{1,n-1}(S_{1,n-1})$, is a rooted product of $S_{1,n-1}$ and $S_{1,n-1}$. To prove that $\eta(S^2_{1,n-1}) = n(n-2)$; let $x_{i,j}$, $i, j = 1,2,\ldots,n$ be a weighting for $S^2_{1,n-1}$, as indicated in Figure 3.5.

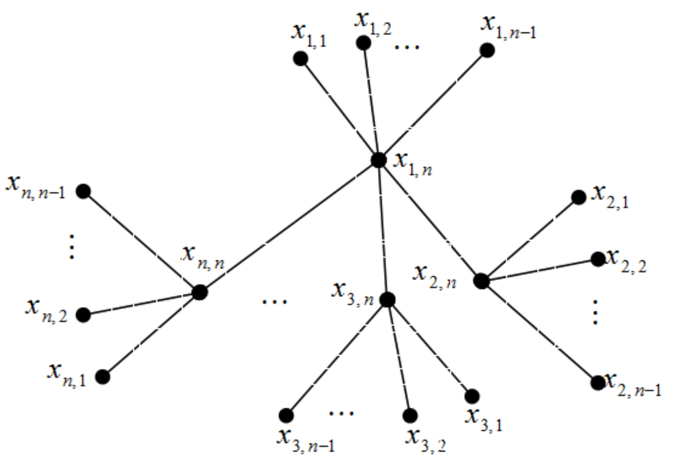


Figure 3.5. A weighting of $S^2_{1,n-1}$, where the rooted vertex of $S_{1,n-1}$ is the central vertex.

Then, from the condition that $\sum_{w \in N_G(v)} f(w) = 0$, for all v in $S^2_{1,n-1}$, we have:

$x_{1,n} = x_{2,n} = \ldots = x_{n,n} = 0$ ...(3.17)

And,

$$x_{1,1}+x_{1,2}+\ldots+x_{1,n-1}=0$$
$$x_{2,1}+x_{2,2}+\ldots+x_{2,n-1}=0$$
$$\vdots$$
$$x_{n,1}+x_{n,2}+\ldots+x_{n,n-1}=0$$

Then,

$$\begin{aligned} x_{1,n-1}&=-x_{1,1}-x_{1,2}-\ldots-x_{1,n-2}\\ x_{2,n-1}&=-x_{2,1}-x_{2,2}-\ldots-x_{2,n-2}\\ &\vdots\\ x_{n,n-1}&=-x_{n,1}-x_{n,2}-\ldots-x_{n,n-2}\end{aligned} \qquad \ldots(3.18)$$

Then, from Equation (3.18), the number of independent variables used in a high zero-sum weighting of $S^2_{1,n-1}$ is equal to $n(n-2)$.

Hence, $\eta(S^2_{1,n-1})=n(n-2)$.

For $k=3$, use the iteration $S^3_{1,n-1}=S^2_{1,n-1}(S_{1,n-1})$, since $S^2_{1,n-1}$ is a dendrimer graph having n star graphs $S_{1,n-1}$ and each graph has $n-1$ vertices to be attachment, thus we attach $S_{1,n-1}$ to $n(n-1)$ vertices. But also, each copy of $S_{1,n-1}$ adds (increases) $(n-3)$ to the nullity of a dendrimer graph, together the variable used in a high zero-sum weighting of $S_{1,n-1}$.

Therefore,

$$\begin{aligned}\eta(S^3_{1,n-1})&=n(n-1)(n-2)+\eta(S_{1,n-1})\\ &=n(n-1)(n-2)+(n-2)\,.\end{aligned}$$

Similarly, we have:

$\eta(S^k_{1,n-1})=n(n-1)^{k-2}\,(n-2)\,+\eta(S^{k-2}_{1,n-1})$, for each k, $k\geq 3$.

ii) The proof is similar to that of Proposition 3.6.■

Corollary 3.9: For a dendrimer graph $S^k_{1,n-1}, k\geq 2$, $n\geq 3$, we have:

i) If k is odd, $k\geq 3$, and the rooted vertex of a graph $S_{1,n-1}$ is its central vertex, then,

$$\eta(S^k_{1,n-1})=(n-2)+n(n-2)\sum_{i=1}^{\frac{k-1}{2}}(n-1)^{2i-1}\,.$$

ii) If k is even, $k\geq 2$, and the rooted vertex of a graph $H=S_{1,n-1}$ is its central vertex, then: $\eta(S^k_{1,n-1})=n(n-2)\sum_{i=0}^{\frac{k-2}{2}}(n-1)^{2i}$.

iii) For all k, $k\geq 2$, if the rooted vertex of a graph $H=S_{1,n-1}$ is a non central vertex, then,

$$\eta(S^k_{1,n-1})=(n-2)+n(n-3)\frac{(n-1)^{k-1}-1}{n-2},\ \text{for all k, } k\geq 2.$$

Proof: i) From Proposition 3.8 (i), we have:

$\eta(S_{1,n-1})=n-2$, $\eta(S^2_{1,n-1})=n(n-2)$, and

$\eta(S^k_{1,n-1})=n(n-1)^{k-2}(n-2)+\eta(S^{k-2}_{1,n-1})$, for all k, $k\geq 3$.

$$\therefore\quad \eta(S^k_{1,n-1})=n(n-1)^{k-2}\,(n-2)\,+n(n-1)^{k-4}\,(n-2)\,+\eta(S^{k-4}_{1,n-1})$$
$$\vdots$$

$$= n(n-1)^{k-2}(n-2) + n(n-1)^{k-4}(n-2) + \ldots + n(n-1)(n-2) + \eta(S_{1,n-1})$$
$$= n(n-1)^{k-2}(n-2) + n(n-1)^{k-4}(n-2) + \ldots + n(n-1)(n-2) + (n-2)$$
$$= n(n-2)\,[(n-1)^{k-2} + (n-1)^{k-4} + \ldots + (n-1)\,] + (n-2)$$
$$= n(n-2)\sum_{i=1}^{\frac{k-1}{2}} (n-1)^{2i-1} + (n-2)\,.$$

$$\therefore \quad \eta(S_{1,n-1}^{\,k}) = (n-2) + n(n-2)\sum_{i=1}^{\frac{k-1}{2}} (n-1)^{2i-1} \text{ , if k is odd, } k \ge 3\,.$$

ii) From Proposition 3.8 (i), we have:

$\eta(S_{1,n-1}^{2}) = n(n-2)$, and

$\eta(S_{1,n-1}^{k}) = n(n-1)^{k-2}(n-2) + \eta(S_{1,n-1}^{k-2})$, for each k, $k \ge 2$

$$\therefore \quad \eta(S_{1,n-1}^{k}) = n(n-1)^{k-2}(n-2) + n(n-1)^{k-4}(n-2) + \eta(S_{1,n-1}^{k-4})$$
$$\vdots$$
$$= n(n-1)^{k-2}(n-2) + n(n-1)^{k-4}(n-2) + \ldots + n(n-1)^{2}(n-2) + \eta(S_{1,n-1}^{2})$$
$$= n(n-1)^{k-2}(n-2) + n(n-1)^{k-4}(n-2) + \ldots + n(n-1)^{2}(n-2) + n(n-2)$$
$$= n(n-2)\,[(n-1)^{k-2} + (n-1)^{k-4} + \ldots + (n-1)^{2} + 1]$$
$$= n(n-2)\sum_{i=0}^{\frac{k-2}{2}} (n-1)^{2i}\ .$$

$$\therefore \quad \eta(S_{1,n-1}^{\,k}) = n(n-2)\sum_{i=0}^{\frac{k-2}{2}} (n-1)^{2i} \text{ , if k is even, } k \ge 2\,.$$

iii) From Proposition 3.8 (ii), we have:

$\eta(S_{1,n-1}) = n-2$, and $\eta(S_{1,n-1}^{k}) = n(n-1)^{k-2}(n-3) + \eta(D^{k-1})$, for all k, $k \ge 2$.

$$\therefore \quad \eta(S_{1,n-1}^{k}) = n(n-1)^{k-2}(n-3) + n(n-1)^{k-3}(n-3) + \eta(S_{1,n-1}^{k-2})$$
$$\vdots$$
$$= n(n-1)^{k-2}(n-3) + n(n-1)^{k-3}(n-3) + \ldots + n(n-1)(n-3) + \eta(S_{1,n-1}^{2})$$

$$= n(n-1)^{k-2}(n-3) + n(n-1)^{k-3}(n-3) + \ldots + n(n-1)(n-3) + n(n-3) + (n-2)$$
$$= n(n-3)\,[(n-1)^{k-2} + (n-1)^{k-3} + \ldots + (n-1) + 1\,] + (n-2)$$
$$= (n-2) + n(n-3)\frac{(n-1)^{k-1}-1}{n-2}\,.$$

$$\therefore \quad \eta(S_{1,n-1}^{\,k}) = (n-2) + n(n-3)\frac{(n-1)^{k-1}-1}{n-2}\text{, for all k, } k \ge 2\,.\blacksquare$$

4. Upper Bounds for the Nullity of Coalescence Graphs

In this section, we shall introduce and prove a lower and an upper bound for the nullity of the coalescence graph $G_1 \circ G_2$.

Proposition 4.1: For any singular graphs G_1 and G_2.

$\eta(G_1) + \eta(G_2) - 1 \le \eta(G_1 \circ G_2) \le \eta(G_1) + \eta(G_2) + 1$

Proof: Let G_1 and G_2 be two singular graphs of orders p_1 and p_2, respectively, thus first we label the vertices of G1 by $u_1, u_2, \ldots u_{p_1}$, with a high zero- sum weighting $x_1, x_2, \ldots, x_{p_1}$ and the vertices of G2 by $v_1, v_2, \ldots, v_{p_2}$, with a high zero- sum weighting $y_1, y_2, \ldots, y_{p_2}$.

Assume that u_1 and v_1 are rooted vertices of G1 and G2 respectively. Then equality holds at the left if either or both rooted vertices are non- zero weighted because there exists a high zero- sum weighting for $G_1 \circ G_2$ which is the enlargement of high zero- sum weightings for G1 and G2 reducing or vanishing one non- zero weight at the identification vertex. See Figure 4.1 where $G_1 = G_2 = P_3$.

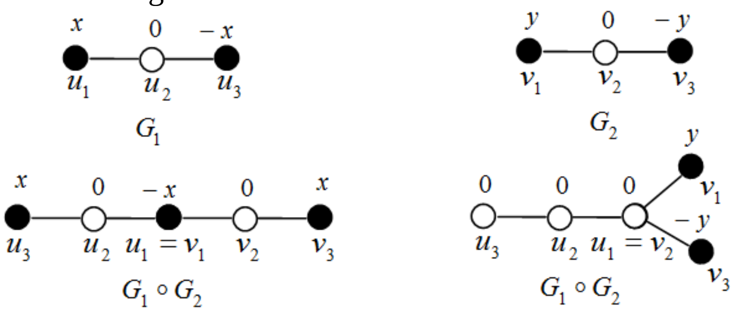
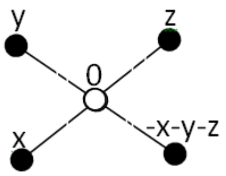


Figure 4.1. $G_1 \circ G_2$ where either or both rooted vertices have non-zero weight.

Moreover, strictly holds at the left side if both rooted vertices have zero weights in their high zero- sum weightings, because there exists a zero- sum weighting which is the union of both high zero-sum weightings of G_1 and G_2.

Equality holds at the right side if both rooted vertices are cut vertices with zero weights in their high zero- sum weightings, and each component obtained with a deleting of a rooted cut vertex is singular, because there exists a high zero- sum weighting for $G_1 \circ G_2$ that uses an extra independent variable further than the variables used in high zero- sum weightings of G_1 and G_2. See Figure 4.2.

Moreover, strictly holds at the right side if one rooted vertices does not satisfy the condition of equality as indicated above. ■



$G_1 \circ G_2$

Figure 4.2. $G_1 \circ G_2$ where both rooted vertices are cut vertices with zero weight and each component obtained by the deleting of rooted vertex is singular.

Note: Let w be the identification vertex $w = (u \equiv v)$ of G $= G_1 \circ G_2$. Then, by interlacing Theorem [2, p314], $|\eta(G) - \eta(G - w)| \leq 1$ i.e
$|\eta(G) - \eta(G_1 - u) - \eta(G_2 - v)| \leq 1 \quad \forall$ u $\in G_1$, v $\in G_2$. Hence,

$$\eta(G_1 - u) + \eta(G_2 - v) - 1 \leq \eta(G_1 \circ G_2) \leq \eta(G_1 - u) + \eta(G_2 - v) + 1.$$

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