On WJCP-Injective Rings

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ABSTRACT

As a generalization of right $JCP$—injective rings, we introduce the nation of right $WJCP$—injective rings, that is for any right nonsingular element $a$ of $R$, there exists a positive integer $n$ and $a^n \neq 0$ and any right $R$—homomorphism $f : a^n R \to R$, there exists $m \in R$ such that $f(a^n c) = m a^n c$ for all $c \in R$. In this paper, we first introduce and characterize a right $WJCP$—injective rings. Next, connection between such ring and quasi $\pi$—regular rings and $S$—weakly regular rings.

Keywords: right $JCP$—injective rings, right $WJCP$—injective rings, quasi $\pi$—regular rings, $S$—weakly regular rings.

Introduction:

Throughout this paper $R$ denotes an associative ring with identity, and $R$—modules are unital. For $a \in R$, $r(a)$ and $l(a)$ denote the right annihilator of $a$ and left annihilator of $a$, respectively. We write $J(R), Y(R), Z(R)$ for the Jacobson radical, the
right singular ideal and the left singular ideal, respectively. An element \( a \in R \) is called right (left) regular if \( r(a)=0 \) (\( l(a)=0 \)).[10]

A ring \( R \) is reduced if \( a^2=0 \) implies \( a=0 \) for all \( a \in R \), and \( R \) is called right \( C_2 \)-ring if every right ideal \( T \) which is isomorphic to summand of \( R \) is a summand [4].

A right \( R \)-module \( M \) is said to be right \( YJ \)-injective [1], if for any \( 0 \neq a \in R \) there exists a positive integer \( n \) such that \( a^n \neq 0 \) and every right \( R \)-homomorphism of \( a^nR \) into \( M \) extended to one of \( R \) into \( M \).

Call a right \( R \)-module \( M \), \( JCP \)-injective, if for each \( k \notin Y(R) \), any right \( R \)-homomorphism \( kR \rightarrow M \) extended to \( R \). Examples of these module include right \( YJ \)-injective modules. The concept of \( JCP \)-injective was first introduced by Wei, [7]. As a generalization of this concept [5], introduced \( WJCP \)-injective as a right \( R \)-module \( M \), \( WJCP \)-injective, if for each \( a \notin Y(R) \), then there exists a positive integer \( n \) such that \( a^n \neq 0 \) and every right \( R \)-homomorphism from \( a^nR \) into \( M \) can be extended to one of \( R \) into \( M \).

A ring \( R \) is called strongly \( \pi \)-regular ring, if for every \( a \in R \) there exists a positive integer \( n \) such that \( a^n = a^{n+1}b \). [2].

A ring \( R \) is called \( S \)-weakly regular ring if for all \( a \in R \) then \( a \in aRa^2R \ (a \in Ra^2Ra) \). [3]

2- \( WJCP \)-Injective Rings:

In this section, some basic properties of \( WJCP \)-injective rings are given.

**Definition 2.1:**[5]

A right \( R \)-module \( M \) is said to be \( WJCP \)-injective if for each \( a \notin Y(R) \) there exists a positive integer \( n \) such that \( a^n \neq 0 \) and every right \( R \)-homomorphism from \( a^nR \) into \( M \) can be extended to one of \( R \) into \( M \). If \( R_k \) is \( WJCP \)-injective ring, we call \( R \) is right \( WJCP \)-injective ring.

Clearly, right \( YJ \)-injective rings are right \( WJCP \)-injective. The ring in Example (2.5,[5]) is a right \( WJCP \)-injective which is not right \( YJ \)-injective.

**Theorem 2.2:**

A ring \( R \) is a right \( WJCP \)-injective if and only if for \( a \notin Y(R) \) there exists a positive integer \( n \) such that \( a^n \neq 0 \) and \( Ra^n = lr(a^n) \).

**Proof:**

Suppose that a ring \( R \) is right \( WJCP \)-injective. Then, for every \( 0 \neq a \notin Y(R) \), there exists a positive integer \( n \) such that \( a^n \neq 0 \) and every right \( R \)-homomorphism from \( a^nR \) into \( R \) can be extended to endomorphism of \( R \). It is clear that \( Ra^n \subseteq l(r(a^n)) \). Let \( x \in l(r(a^n)) \), and \( f : a^nR \rightarrow R \) are defined by \( f(a^nr) = xr \), then, \( f \) is well defined right \( R \)-homomorphism because \( xr(a^n) = 0 \) So \( r(a^n) \subseteq r(x) \).

Since, \( R \) is right \( WJCP \)-injective, there exists \( c \in R \) such that \( f(a^n) = ca^n \). Then, \( x = f(a^n) = ca^n \in Ra^n \) which implies that \( lr(a^n) \subseteq Ra^n \). Consequently, \( lr(a^n) = Ra^n \)
Conversely, let \( a \not\in Y(R) \) there exists a positive integer \( n \) such that \( Ra^n = lr(a^n) \). Let \( f : a^nR \to R \) be any right \( R \) - homomorphism. Then, \( r(a^n) \subseteq r(f(a^n)) \), which implies \( f(a^n) \in lr(f(a^n)) \subseteq lr(a^n) = Ra^n \) and therefore \( f(a^n) = da^n \) for some \( d \in R \), this shows that \( R \) is right \( WJCP \)-injective. 

**Example: [7, Example 2.4]**

Let \( V \) be a two- dimensional vector space over a field \( F \), the trivial extension \( R = T(F,V) = F \oplus V \) is commutative, local, artinian ring with \( J^2 = 0 \) and \( J(R) = Y(R) \). Now, if \( x \in R \) with \( x \notin Y(R) \), then \( x \) is invertible. So, \( l(r(x^n)) = R = R x^n \). This implies that \( R \) is right \( WJCP \)-injective.

**Proposition 2.3:**

Let \( R \) be a right \( WJCP \)-injective and \( Ra^n \subseteq Ra \) for all \( a \in R \) and a positive integer \( n \). Then, any right regular element of \( R \) is left invertible.

**Proof:**

Let \( a \in R \) and there exists a positive integer \( n \) such that \( r(a^n) = 0 \). Since, \( R \) is right \( WJCP \)-injective ring, then \( R = lr(a^n) = Ra^n \subseteq Ra \) by Theorem 2.2. In particular \( ra = 1 \) for some \( r \in R \). Hence, \( a \) is left invertible.

Wei and Chen [5] proved the following theorem:

**Theorem 2.4:**

Let \( R \) be right \( WJCP \)-injective ring. Then,

1. \( Y(R) \subseteq J(R) \)
2. \( R \) is a right \( C_2 \)-ring.

Following [6], a right \( R \)-module is called \( N \)-flat if for each \( a \in N(R) \) then, the mapping \( I_M \otimes i : M \otimes_kRa \to M \otimes_kR \) is monic, where \( i : Ra \to R \) is the inclusion map.

**Lemma 2.5:** [6]

Let \( I \) be a right ideal of \( R \). Then, \( R/I \) is \( N \)-flat right \( R \)-module if and only if \( Ia = I \cap Ra \) for all \( a \in N(R) \).

**Theorem 2.6:**

If \( R \) is a right \( WJCP \)-injective, \( l(a) \subseteq r(a) \) for every \( a \in R \) and every simple singular right \( R \)-module is \( N \)-flat, then \( Z(R) = 0 \).

**Proof:**

If \( Z \neq 0 \), then \( 0 \neq b \in Z \) such that \( b^2 = 0 \). We show that \( Z + r(b) = R \). Otherwise there exists a maximal right ideal \( M \) such that \( Z + r(b) \subseteq M \). If \( M \) is not an essential right ideal of \( R \), then \( M = r(e) \) where \( e^2 = e \in R \). If \( be \neq 0 \), then \( beR \supseteq eR \) as right \( R \)-module by Theorem 2.4 (\( R \) is \( C_2 \)-ring). \( beR = gR \), where \( g^2 = g \in R \) so, \( g \in Z \) because \( beR \subseteq Z \). This is a contradiction. So, \( be = 0 \). Then \( e \in r(b) \subseteq M = r(e) \) which
is impossible. Thus, $M$ is an essential right ideal of $R$, so $R/M$ is $N$–flat, by Lemma 2.5, $b = ab$ for some $a \in M$, so $1 - a \in r(b) \subseteq M$ and then, $1 \in M$, which is a contradiction. Hence, $Z + r(b) = R$, let $1 = x + y$, $x \in Z$, $y \in r(b)$ then, $b = bx$ and so $b(1 - x) = 0$ since $x \in Z$ and $l(x) \cap l(1 - x) = 0$, $l(1 - x) = 0$, hence $b = 0$ which is a contradiction so $Z(R) = 0$. #

3- The Connection between WJCP – Injective Rings and other Rings.

In this section, we give the relation between WJCP – injective, $S$ – weakly regular rings, strongly $\pi$ – regular rings.

Following [7], a ring $R$ is called right quasi regular if $a \in aRa$ for all $a \in Y(R)$. Now, we give the generalized of quasi regular ring.

**Definition 3.1:**

A ring $R$ is called right quasi $\pi$ – regular rings if $a^n \in a^nRa^n$ for all $a \in Y(R)$ and a positive integer $n$. Clearly $R$ is $\pi$ – regular ring if and only if $R$ is right non singular and right quasi $\pi$ – regular.

**Example:**

Let $Z_6$ be a ring of integers modulo 6, then $Y(R) = \{0\}$ so for all $a \in Y(R)$, there exists a positive integer $n$ such that $a^n = a^nRa^n$.

**Proposition 3.2:**

The following conditions are equivalent for a ring $R$:

1- $R$ is right quasi $\pi$ – regular ring.

2- Every $R$ – module is WJCP – injective.

3- Every cyclic $R$ – module is WJCP – injective.

**Proof:**

1 $\rightarrow$ 2:

Let $M$ be an $R$ – module, $a \in R$ with $a \notin Y(R)$ and $f : a^nR \rightarrow M$ any right $R$-homomorphism. Since, $R$ is right quasi $\pi$ – regular rings $a^n = a^nba^n$ for some $b \in R$. Let $a^n = e$ and $f(e) = m$, where $m \in M$. Then, $g : R \rightarrow M$ is defined by $g(r) = mr$, $r \in R$ is a right $R$-homomorphism, and $g(a^n) = ma^n = f(e)a^n = f(a^nba^n)r = f(a^n)a^n r = f(a^n)r = f(a^n)r$ which implies that $M$ is WJCP – injective.

2 $\rightarrow$ 3:

is trivial.

3 $\rightarrow$ 1:

Let $a \notin Y(R)$. Since, $a^nR$ is WJCP – injective, then the identity map $a^nR \rightarrow a^nR$ can be extended to one of $R$ into $R$. Hence, $a^n = a^nba^n$ for some $b \in R$. Thus, $R$ is right quasi $\pi$ – regular ring. #

A ring $R$ right weakly principally small injective [5], if for any $a \neq 0 \in J(R)$, there exists a positive integer $n$ such that $a^n \neq 0$ and any $R$–homomorphism from $a^nR$ to $R_a$ can be extended to $R_a$ into $R_a$. Clearly, every right $YJ$ – injective is right weakly principally small injective.
The following theorem is a generalization of [7, Theorem 2.9].

Theorem 3.3:

$R$ is right $YJ$-injective if and only if $R$ is right WJCP-injective and right weakly principally small injective.

Proof:

Assume $R$ is $YJ$-injective, then $R$ is right WJCP-injective and weakly principally small injective. Conversely, let $R$ be a right WJCP-injective, then by Theorem 2.4, $Y(R) \subseteq J(R)$.

Let $a \in R$. If $a \notin Y(R)$, then by Theorem 2.2., then $a \notin Y(R)$. If $l(r(a^n)) = Ra^n$. Then $x \in l(r(a^n))$ is clear. Let $Ra^n \subseteq l(r(a^n))$. $Ra^n = l(r(a^n))$ we claim that $a \in J(R)$ is a well defined $f$. Then, $f(a^nr) = xr$ be defined by $f : a^nR \to R$. Let $r(a^n) \subseteq r(x)$ right $R$-homomorphism. Since $R$ is right weakly principally small injective, there exists a right $R$-homomorphism $g : R \to R$ such that $f(a^n) = g(a^n)$. Hence, $x = f(a^n) = g(a^n) = g(1)a^n \in Ra^n$ and so $l(r(a^n)) \subseteq Ra^n$, hence $Ra^n = l(r(a^n))$ therefore, $R$ is $YJ$-injective. #

Lemma 3.4:[8]

If $R$ is $S$-weakly regular ring if and only if $R$ is reduced weakly regular ring.

Now, we have the following theorem:

Theorem 3.5:

Let $R$ be a ring whose simple singular right $R$-modules are WJCP-injective. Then, $R$ is reduced if and only if $R$ is $S$-weakly regular ring.

Proof:

If $R$ is $S$-weakly regular ring then, $R$ is reduced by Lemma 3.4. Conversely, assume that $R$ is reduced. For any $0 \neq a \in R$, if $Ra^2R + r(a) \neq R$, then there exists a maximal right ideal $M$ of $R$ containing $Ra^2R + r(a)$. If $M$ is not an essential right ideal in $R$, then $M = r(e), e^2 = e \in R$. Therefore, $ea = 0$, since $R$ is abelian, $ae = 0$ hence, $e \in r(a) \subseteq M = r(e)$, which is a contradiction. So, $M$ is an essential right ideal in $R$ by hypothesis, $R/M$ is WJCP-injective. Since, $R$ is reduced, $Y(R) = 0$. Hence, there exists a positive integer $n$ such that $a^{2n} \neq 0$ and any right $R$-homomorphism $a^{2n}R \to R/M$ can be extended to $R \to M$. Set $f : a^{2n}R \to R/M$ is defined by $f(a^{2n}x) = x + M, x \in R$. Then, $f$ is a well defined right $R$-homomorphism. Hence, there exists $g : R \to R/M$ such that $1 + M = f(a^{2n}) = g(a^{2n}) = g(1)a^{2n} = ca^{2n} + M$ where, $g(1) = c + M$, so $1 - ca^{2n} \in M$. Since, $ca^{2n} \in Ra^2R \subseteq M$, $1 \in M$ which is a contradiction. Hence, $Ra^2R + r(a) = R$. In particular, $ca^2d + x = 1$, for some $c, d \in R$, $x \in r(a)$, then $a = aca^2d$. Therefore, $R$ is $S$-weakly regular rings. #

Definition 3.6:[9]
$R$ is called right $\text{CAM} − \text{ring}$, if for any maximal essential right ideal $M$ of $R$ (if it exists) and for any right sub ideal $I$ of $M$ which is either a complement right sub ideal of $M$ or a right annihilator ideal in $R$, $I$ is an ideal of $M$.

Show that semi prime right $\text{CAM} − \text{ring}$, $R$ is either semi simple artinian or reduced.

**Lemma 3.7:**[9]

If $R$ is a semi prime right $\text{CAM} − \text{ring}$ then, $R$ is either semi simple artinian or reduced.

**Theorem 3.8:**

Let $R$ be a semi prime right $\text{CAM} − \text{ring}$, quasi duo ring whose simple singular right $R −$ modules are $\text{WJCP} − \text{injective}$. Then $R$ is strongly $\pi − \text{regular ring}.$

**Proof:**

If $R$ is not a semi simple artinian ring then, $R$ is reduced so $R$ is a right non singular ring. Let $0 \neq a \in R$. If $a^n R + r(a^n) \neq R$, then there exists a maximal right ideal $M$ of $R$ such that $a^n R + r(a^n) \subseteq M$. If $M$ is not an essential right ideal of $R$, then $M = r(e)$ where $e = e^2 \in R$ because $R$ is reduced $ea = ae = 0$ and $e \in r(a) \subseteq M = r(e)$ is contradiction. Hence, $M$ is an essential right ideal of $R$ and so $R/M$ is a singular simple right $R −$ module. By hypothesis $R/M$ is right $\text{WJCP} − \text{injective}$. Then, there exists $c \in R$ such that $1 − ca^n \in M$. But, then $1 \in M$ because $R$ is a quasi duo ring and $M$ is an ideal. It is contradiction. Hence, $a^n R + r(a^n) = R$ and $R$ is a strongly $\pi − \text{regular ring}.$

We conclude the paper with a few characteristic properties of $\text{WJCP} − \text{injective}$ ring.

**Proposition 3.9:**

Let $R$ be a reduced ring and every left principle ideal is a left annihilator of an element in $R$. Then, the followings are

1- $R$ is strongly regular.
2- $R$ is right $\text{YJ} − \text{injective}.$
3- $R$ is $\text{WJCP} − \text{injective}.$
4- $R$ is simple $\text{WJCP} − \text{injective}.$
5- $R$ is simple singular $\text{WJCP} − \text{injective}.$
6- $R$ is $S − \text{weakly regular ring}.$

**Proof:**

$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5$, $6 \rightarrow 1$ are trivial

$5 \rightarrow 6$ : by Theorem 3.5. #
REFERENCES


