The Detour Polynomials of the Corona of Graphs

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ABSTRACT

A new graph distance based polynomial, called detour polynomial, is introduced. The detour polynomial and the detour index of the corona $G_1 \circ G_2$ of two connected disjoint graphs $G_1$ and $G_2$ are obtained in this paper.

Keywords: distance, detour distance, detour polynomial, detour index, corona.

1. Introduction:

The concept of Hosoya polynomial was first put forward in 1988 by Hosoya \cite{14}. Several authors, such as \cite{1}-\cite{4}, \cite{12}, and \cite{14}-\cite{17} had obtained Hosoya polynomials for special graphs, graphs having some kind of regularity and for compound graphs obtained by using some well-known binary operations in graph theory.

In this paper, we consider finite connected graphs without loops or multiple edges. For undefined concepts and notations see \cite{6} and \cite{13}.

The standard distance $d(u,v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u-v$ path in $G$. This is not the only way, however, that distance has been defined on the vertex set of a connected graph. The length of a longest $u-v$ path between two vertices $u$ and $v$ in a connected graph is called the detour distance $D(u,v)$ between $u$ and $v$. As with standard distance, detour distance is a metric on the vertex set of any connected graph \cite{7}. A $u-v$ path of length $D(u,v)$ is $u-v$ detour. It is clear that $D(u,v)=1$ if and only if $uv$ is a bridge of $G$, and $D(u,v)=p(G)-1$ if and only if $G$ contains a Hamiltonian $u-v$ path. Furthermore,
\[ d(u,v) = D(u,v) \] for every two vertices of a graph \( G \). If it is possible, however, that \( d(u,v) = D(u,v) \) for some pairs of distinct vertices in a graph that contains no bridges. The **detour eccentricity** \( e_D(v) \) of a vertex \( v \) is the maximum detour distances from \( v \) to all other vertices in \( G \). The **detour radius** \( rad_D(G) \) of a connected graph \( G \) is the minimum detour eccentricity among the vertices of \( G \), and the **detour diameter** \( diam_D(G) \) is the maximum detour eccentricity among the vertices of \( G \). Since \( d(x,y) \leq D(x,y) \) for every two vertices \( x \) and \( y \) in \( G \), it follows that \( e(v) \leq e_D(v) \) for every vertex \( v \). Therefore, \( rad(G) \leq rad_D(G) \) and \( diam(G) \leq diam_D(G) \) for every connected graph \( G \) (see [9], [10] and [18]). In fact \( rad_D(G) \leq diam_D(G) \leq 2rad_D(G) \), see [9].

Let \( G \) be a connected graph of order \( p \) and size \( q \). The **detour polynomial** of \( G \), denoted by \( D(G;x) \), is defined by

\[
D(G;x) = \sum_{u,v \in V} x^{D(u,v)} , \; u \neq v ; \; V = V(G). \]

It is clear that \( D(G;x) = \sum_{k \geq 1} C_D(G,k)x^k \), in which \( C_D(G,k) \) is the number of unordered pairs \( u,v \) such that \( D(u,v) = k \).

Amić and Trinajstić [5] were first to consider the **detour index** \( dd(G) \) defined as the sum of the detour matrix elements above the main diagonal. The detour index is used in quantitative structure-activity relationship (QSAR) studies. Lukovits [15] tested the detour index on the correlation of the boiling points of alkanes of cycloalkanes. The detour index can also be obtained from the detour polynomial, because

\[
dd(G) = \sum_{u,v} D(u,v) = \left. \frac{d D(G;x)}{dx} \right|_{x=1} = \sum_{k \geq 1} kC_D(G,k). \]

Let \( C_D(G;v,k) \) be the number of vertices \( u (\neq v) \) such that \( D(u,v) = k \). Then, we define for each vertex \( v \) of \( G \):

\[
D(G;v,x) = \sum_{k \geq m(v)} C_D(G;v,k)x^k = \sum_{u \in V} x^{D(v,u)},
\]

where \( e_D(v) \) is the detour eccentricity of vertex \( v \), and \( m(v) \) is the minimum detour distance from \( v \). This polynomial is called the **detour polynomial of vertex** \( v \). It is clear that

\[
D(G;x) = \frac{1}{2} \sum_{v \in V} D(G;v,x). \]

2. The Corona \( G_1 \odot G_2 \):

The **corona** of two disjoint graphs \( G_1 \) and \( G_2 \) of orders \( p_1 \) and \( p_2 \), respectively, is the graph \( G = G_1 \odot G_2 \) defined by taking one copy of \( G_1 \) and \( p_1 \) copies of \( G_2 \), and then joining the \( i \)th vertex of \( G_1 \) to every vertex in the \( i \)th copy of \( G_2 \), as illustrated in Fig.2.1, where the copies of \( G_2 \) are denoted by \( G'_1, G'_2, \ldots, G'_{p_2} \),

\[
V'_i = V(G'_i) = \{ v_1, v_2, \ldots, v_{p_2} \},
\]

\[
U^{(i)} = V(G'_i) = \{ u_1^{(i)}, u_2^{(i)}, \ldots, u_{p_1}^{(i)} \}, \text{ for } i = 1,2,\ldots, p_1.
\]
The Detour Polynomials of the Corona of Graphs

\[ V_2 = V(G_2) = \{u_1, u_2, \ldots, u_{p_2}\}, \]
and
\[ U = \bigcup_{i=1}^{p_1} U^{(i)} . \]

![Diagram](image.png)

**Fig. 2.1 The corona \( G_1 \circ G_2 \)**

It follows, from the definition of the corona \( G_1 \circ G_2 \), that
\[ p(G_1 \circ G_2) = p_1(p_1 + p_2), \quad q(G_1 \circ G_2) = q_1 + p_1q_2 + p_1p_2 \]
and \( \text{diam}(G_1 \circ G_2) = \text{diam}(G_1) + 2 \).

Note that \( G_1 \circ G_2 \neq G_2 \circ G_1 \) unless \( G_1 \cong G_2 \).

Thus, the corona is a binary graph operation, it was defined for the first time by Frucht and Harary [11] in 1970, and used in studying the automorphism group of graphs. Recently, in 2007, the Hosoya polynomial of the corona of two graphs, with respect to Steiner distance, was obtained [1]. So, we think that it is an interesting to obtain the Hosoya polynomial of the corona with respect to detour distance.

We begin discussing the detour distance and detour diameter of \( G_1 \circ G_2 \). Let \( e_D^{(i)}(w) \) be the detour eccentricity of vertex \( w \) in the graph \( G^{(i)} \), \( i=1,2 \). Define the graph \( G^{(i)}_2 \) as \( G_2 + K_1 \).

**Proposition 2.1:** Let \( w_1 \) and \( w_2 \) be any two distinct vertices of \( G = G_1 \circ G_2 \). Then, the detour distance \( D_G(w_1, w_2) \) equals:

1) \( D_G(w_1, w_2) \), if \( w_1, w_2 \in V(G_1) \);
2) \( D_{G_1}(w_1, w_2) \), if \( w_1, w_2 \in V(G_1') \) for \( 1 \leq i \leq p_1 \);
3) \( 1 + e_D^{(2)}(w_2) \), if \( w_i = v_i \) and \( w_2 \in V(G_i') \) for some \( i, 1 \leq i \leq p_1 \);
4) \( D_{G_i}(v_i, v_j) + e_D^{(2)}(w_2) + 1 \), if \( w_1 = v_i \) and \( w_2 \in V(G_i') \), where \( i \neq j \) and \( 1 \leq i, j \leq p_1 \);
5) \( D_{G_i}(v_i, v_j) + e_D^{(2)}(w_1) + e_D^{(2)}(w_2) + 2 \), if \( w_1 \in V(G_i'), w_2 \in V(G_j') \), where \( i \neq j \) and \( 1 \leq i, j \leq p_1 \).

**Proof:**
1) Obvious.
2) Any detour between two vertices of \( G_i' \), \( 1 \leq i \leq p_1 \) must contain vertex \( v_i \) and does not contain another vertex of \( G_1 \). Therefore \( D_G(w_1, w_2) = D_{G_i}(w_1, w_2) \).
3) If $w_2'$ is the detour eccentric vertex of $w_2$ in $G'_i$, that is $D_{G_i}(w_2, w_2') = e^{(2)}_{D}(w_2)$, then a detour from $v_i$ to $w_2$ consists of the edge $v_i w_2'$ followed by the detour in $G'_i$ between $w_2'$ to $w_2$. Therefore $D_G(w_1, w_2) = 1 + e^{(2)}_{D}(w_2)$.

4) In this case a detour between $w_1$ and $w_2$ in $G$ consists of a detour between $v_i$ and $v_j$ in $G_i$ followed by the edge $v_i w_2'$ then the detour between $w_2'$ and $w_2$ in $G'_j$, here $w_2'$ is the detour eccentric vertex of $w_2$ in $G'_j$. Thus

$$D_G(w_1, w_2) = D_{G_i}(v_i, v_j) + 1 + e^{(2)}_{D}(w_2).$$

5) A detour between a vertex $w_1$ of $G_i'$ and a vertex $w_2$ of $G'_j$, $i \neq j$, is constructed from a detour from $w_1$ to $w'_1$ (the detour eccentric vertex of $w_1$), followed by the edge $w'_1 v_j$, then a detour from $v_j$ to $v_j$ in $G_i$, then followed by the edge $v_j w_2'$ and finally followed by a detour $w_2' - w_2$ in $G'_j$. The length of this detour is $D_{G_i}(v_i, v_j) + e^{(2)}_{D}(w_1) + e^{(2)}_{D}(w_2) + 2$. Hence, the proof is complete. ■

**Proposition 2.2:** For a connected nontrivial graph $G_i$ and any connected graph $G_2$, $diam_D(G_i \circ G_2) = diam_D(G_i) + 2diam_D(G_2) + 2$.

**Proof:** Let $v_i, v_j$ be two vertices of $G_i$ such that $D_{G_i}(v_i, v_j) = diam_D(G_i)$; and let $u_1, u_2$ be two vertices of $G_2$ such that $D_{G_2}(u_1, u_2) = diam_D(G_2)$.

One may easily check from Fig.2.1 and Proposition 2.1(5), that

$$D_G(u_i^{(i)}, u_k^{(j)}) = D_{G_i}(u_i^{(i)}, u_k^{(i)}) + 1 + D_{G_i}(v_i, v_j) + 1 + D_{G_j}(u_i^{(j)}, u_k^{(j)})$$

$$= diam_D(G_i) + 2diam_D(G_2) + 2.$$

Now, let $w_1, w_2$ be any two vertices of $G_i \circ G_2$, then:

1- If $w_1, w_2 \in V(G_i)$, then $D_G(w_1, w_2) \leq diam_D(G_i)$;

2- If $w_1, w_2 \in V(G'_i)$ for some $r$, $1 \leq r \leq p_i$, then, let $Q$ be a $w_1 - w_2$ detour. If $v_r \not\in V(Q)$, then $D_G(w_1, w_2) \leq diam_D(G_2)$.

If $v_r \in V(Q)$, then $Q - v_r$ consists of two paths in $G_i$ each of length not more than $diam_D(G_2)$. Thus $D_G(w_1, w_2) \leq 2diam_D(G_2) + 2$;

3- If $w_1 \in V(G_i')$ and $w_2 \in V(G'_i)$, $t \neq s$, then

$$D_G(w_1, w_2) \leq diam_D(G_i) + 2diam_D(G_2) + 2;$$

4- If $w_1 \in V(G_i')$ and $w_2 \in V(G'_m)$, then

$$D_G(w_1, w_2) \leq diam_D(G_i) + 1 + diam_D(G_2).$$

Therefore, for all cases of $w_1$ and $w_2$, we have

$$D_G(w_1, w_2) \leq diam_D(G_i) + 2diam_D(G_2) + 2.$$ Hence, the proof is complete. ■

The **minimum detour** of a connected graph $G$ is denoted by $D_{\min}(G)$, and defined as

$$D_{\min}(G) = \min\{D(u, v): u \neq v, u, v \in V(G)\}.$$
Let $H_1$ and $H_2$ be disjoint connected graphs, and let $u_1$ and $u_2$ be vertices of $H_1$ and $H_2$, respectively. Then, the vertex identified graph $H_1 \bullet H_2$ is obtained from $H_1$ and $H_2$ by identifying the vertices $u_1$ and $u_2$. We notice, that

\[ D_{\text{min}}(H_1 \bullet H_2) = \min\{D_{\text{min}}(H_1), D_{\text{min}}(H_2)\} . \]

Applying this fact, we can easily prove the following proposition, which determines $D_{\text{min}}(G_1 \circ G_2)$.

**Proposition 2.3**: For disjoint connected nontrivial graphs $G_1$ and $G_2$, we have

\[ D_{\text{min}}(G_1 \circ G_2) = \min\{D_{\text{min}}(G_1), D_{\text{min}}(G_2^+)\} . \]

3. The Detour Polynomial of $G_i \circ G_j$:

To determine the detour polynomial of the corona $G_i \circ G_j$, we introduce the detour eccentric polynomial of a connected graph $G$, defined as follows:

\[ D_{\text{ecc}}(G; x) = \sum_{v \in V(G)} x^{\text{ecc}_v} . \]

For example:

\[ D_{\text{ecc}}(C_p; x) = px^{p-1} \]

\[ D_{\text{ecc}}(P_p; x) = \begin{cases} 
2 \sum_{i=1}^{p} x^{p-i}, & \text{for even } p \\
\frac{p-1}{2} x^{p-1} + 2 \sum_{i=1}^{p-1} x^{p-i}, & \text{for odd } p. 
\end{cases} \]

We shall obtain the detour polynomial of the corona $G_i \circ G_j$ in terms of $D(G_i; x)$ and $D_{\text{ecc}}(G_j; x)$, $i = 1, 2$.

We set the following definitions:

\[ D(G,V; x) = \sum_{i=1}^{p_i} D(G_i,v_i; x) , \]

\[ D(G,U^{(i)}; x) = \sum_{j=1}^{q_i} D(G_i,u^{(i)}_j; x) , \]

\[ D(G,U; x) = \sum_{i=1}^{p_i} D(G_i,U^{(i)}; x) . \]

Then

\[ D(G; x) = \frac{1}{2} [D(G,V; x) + D(G,U; x)] . \]

**Proposition 3.1**:  

\[ D(G,V; x) = 2D(G; x) + xp_1D_{\text{ecc}}(G_2; x) + 2xD_{\text{ecc}}(G_2; x)D(G_1; x) . \]

**Proof**: From Proposition 2.1, for each $v_i \in V(G_i)$, we have

\[ D(G, v_i; x) = \sum_{j=1}^{q_i} x^{D(v_i,u_j)} + \sum_{\text{wtd}(v_i,w)} x^{D(v_i,w)} + \sum_{\text{wtd}(v_i,w)} x^{D(v_i,w)} . \]

Summing over all $i=1, 2, \ldots, p_1$, we get
**Proposition 3.2:**

\[ D(G,U;x) = 2p_1D(G_2^+;x) - p_1xD_{\text{ecc}}(G_2;x) + xD_{\text{ecc}}(G_2;x).2D(G_1;x) \]

\[ + x^2[D_{\text{ecc}}(G_2;x)]^2.2D(G_1;x). \]

**Proof:** From Proposition 2.1, for each \( u_i^{(j)} \), \( i = 1, 2, ..., p_2 \) and \( j = 1, 2, ..., p_1 \), we have

\[ D(G,u_i^{(j)};x) = D(G_2^+,u_i^{(j)};x) + \sum_{k=1}^{p_i} x^{D_1(\nu_i,\nu_j)v} + \sum_{n=1,k=1}^{p_i,p_j} x^{2\text{ecc}(u_i,v_j)v} + D_2(\nu_i,\nu_j)v. \]

Summing over both \( j = 1, 2, ..., p_1 \) and \( i = 1, 2, ..., p_2 \), we get, using the above notations and definitions:

\[ D(G,U;x) = p_1[2D(G_2^+;x) - xD_{\text{ecc}}(G_2;x)] + xD_{\text{ecc}}(G_2;x).2D(G_1;x) \]

\[ + x^2D_{\text{ecc}}(G_2;x).D_{\text{ecc}}(G_2;x).2D(G_1;x). \]

Hence, the proof is complete. ■

**Theorem 3.3:** For connected nontrivial graphs \( G_1 \) and \( G_2 \), the detour polynomial of the corona graph \( G_1 \odot G_2 \) is

\[ D(G_1 \odot G_2;x) = [1 + xD_{\text{ecc}}(G_2;x)]^2.D(G_1;x) + p_1D(G_2^+;x). \]

**Proof:** From Propositions 3.1 and 3.2, we obtain

\[ D(G_1 \odot G_2;x) = \frac{1}{2}[2D(G_1;x) + p_1xD_{\text{ecc}}(G_2;x) + 2xD_{\text{ecc}}(G_2;x)D(G_1;x) \]

\[ + x^2D_{\text{ecc}}(G_2;x)[D_{\text{ecc}}(G_2;x)]^2] \]

\[ = [1 + xD_{\text{ecc}}(G_2;x)]^2.D(G_1;x) + p_1D(G_2^+;x). \] ■

To obtain the detour index of the corona \( G_1 \odot G_2 \), we give the definition of the **eccentricity sum of a graph**, denoted by \( S_{\text{ecc}}(G) \):

\[ S_{\text{ecc}}(G) = \sum_{v \in V(G)} \text{ecc}_D(v). \]

**Corollary 3.4:** The detour index of \( G_1 \odot G_2 \) is given as

\[ dd(G_1 \odot G_2) = (1 + p_2)^2.dd(G_1) + p_1(p_1 - 1)(p_2 + 1)[p_2 + S_{\text{ecc}}(G_2)] + p_1.dd(G_2^+). \]

**Proof:** Taking the derivative of \( D(G_1 \odot G_2;x) \) given in Theorem 3.3, with respect to \( x \), we get

\[ D'(G_1 \odot G_2;x) = [1 + xD_{\text{ecc}}(G_2;x)]^2.D'(G_1;x) \]

\[ + 2[1 + xD_{\text{ecc}}(G_2;x)][D_{\text{ecc}}(G_2;x) + xD_{\text{ecc}}(G_2;x)].D(G_1;x) \]

\[ + p_1D'(G_2^+;x) \]
Then, putting $x = 1$, we get
\[ dd(G_1 \circ G_2) = D'(G_1 \circ G_2; 1) \]
\[ = (1 + p_2)^2 \, dd(G_1) + 2(1 + p_2)(p_2 + D'_{ec}(G_2; 1)) \cdot \frac{1}{2} \, p_1(p_1 - 1) + p_1 dd(G_2^+) \]

Then, simplifying the above expression and noticing that
\[ D'_{ec}(G_2; 1) = S_{ec}(G_2) \]
we get the required formula of $dd(G_1 \circ G_2)$.

**Example:** Let $G_1 = C_n$, the cycle of order $n$, and $G_2 = K_m$, the complete graph of order $m$, then
\[ D(C_n; x) = \begin{cases} n(x^{n-1} + x^{n-2} + ... + x^{\frac{n}{2}}), & \text{for odd } n, \\ n(x^{n-1} + x^{n-2} + ... + x^{\frac{n}{2}} + \frac{1}{2} x^{\frac{n}{2}}), & \text{for even } n, \end{cases} \]
\[ D_{ec}(K_m; x) = mx^{m-1}, \]
and
\[ D(G_2^+; x) = D(K_{m+1}; x) = \frac{1}{2} m(m+1)x^m. \]

Thus, using Theorem 3.3, we get
\[ D(C_n \circ K_m; x) = \frac{1}{2} nm(m+1)x^m \]
\[ + (1 + mx^m)^2 \cdot \begin{cases} n(x^{n-1} + x^{n-2} + ... + x^{\frac{n}{2}}), & \text{for odd } n \\ n(x^{n-1} + x^{n-2} + ... + x^{\frac{n}{2}} + \frac{1}{2} x^{\frac{n}{2}}), & \text{for even } n. \end{cases} \]
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