On Simple Singular N-Flat Modules

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ABSTRACT

Let I be a right ideal of a ring \( R \), then \( R/I \) is right N-flat module if and only if for each \( a \in I \), there exists \( b \in I \) and a positive integer \( n \) such that \( a^n \neq 0 \) and \( a^n = ba^n \). In this paper, we first introduce and characterize rings whose every simple singular right \( R \)-module is N-flat. Next, we investigate the strong regularity of rings whose every simple singular right \( R \)-module is N-flat. It is proved that:

- \( R \) is strongly regular ring if and only if \( R \) is a wjc , MERT and 2 - primal ring whose simple singular right \( R \)-module is N-flat.

Let \( R \) be a wjc ring satisfying condition (*). If every simple singular right \( R \)-module is N-flat. Then, the Center of \( R \) is a regular ring.

Keywords: N-Flat, MC2 - ring, WPSI ring, GQ - injective, CAM - ring
1. Introduction

Throughout this paper, R denotes an associative ring with identity, and all modules are unitary right R-module. For any nonempty subset X of a ring R, \( r(X) \) and \( l(X) \) denote the right annihilator of X and the left annihilator of X, respectively. If \( X = \{a\} \), we use the abbreviation \( l(a) \) and \( r(a) \). We write \( J(R), Z(R) \) and \( Y(R) \) for the Jacobson radical of R, the left singular ideal of R and the right singular ideal of R, respectively.

Recall that, Let I be a right ideal of a ring R, then \( R/I \) is a right flat R-module if and only if for each \( a \in I \), there exists \( b \in I \) such that \( a = ba \) (cf. [7],[8],[3]). The generalization of flat module to N-flat module is performed as follows: Let I be a right ideal of a ring R, then \( R/I \) is a right N-flat module if and only if for each \( a \in I \), there exists \( b \in I \) and a positive integer \( n \) such that \( a^n \neq 0 \) and \( a^n = ba^n \) [1]. And, in [1], we give a lot of characterizations of right N-flat. For example, \( J(R) \cap Y(R) = (0) \), if \( R \) satisfies condition (*) whose every simple singular right R-module is N-flat.

The ring \( R \) is said to be 2-primal if \( N(R) = P(R) \), where \( N(R) \) is the set of all nilpotent element and \( P(R) \) is the prime radical of \( R \) [4]. The ring \( R \) is called right MC2 ring, if \( aRa = 0 \) for \( a,b \in R \), implies \( bRa = 0 \) [12]. The ring \( R \) is said to be reduced if \( R \) has no nonzero nilpotent element. The ring \( R \) is called right SXM [12], if for each \( 0 \neq a \in R \), \( r(a) = r(a^n) \) for all a positive integer \( n \) satisfying \( a^n \neq 0 \). For example, reduced rings are right SXM rings. The concept of regular rings was introduced in (1936) by Von Neumann [9]. The ring \( R \) is called MERT, if every essential maximal right ideal of \( R \) is an ideal. The ring \( R \) is called regular (strongly regular) ring, if for every \( a \in R \), \( R + aRa \). The ring \( R \) is called right weakly regular ring, if for every \( a \in R \), \( a \in aRaR \) [7].

2. Characterizations of Simple Singular R-Module is N-Flat

This section is devoted to study rings whose every simple singular right R-module is N-flat with some of their basic properties. On the other hand, we characterized MC2 rings terms of simple singular right R-modules is N-flat.

Definition 2.1 [10]

The ring \( R \) is called right MC2 ring, if \( eRe = 0 \) implies \( aRe = 0 \), where \( a \in R \), \( e^2 = e \in R \) and \( K \cong eR \) is minimal right ideal of \( R \), or equivalently if \( eRe \) are minimal, \( e^2 = e \in R \); then \( K = gR \) for some \( g^2 = g \in R \).

We consider the condition (*) ; \( R \) satisfies \( l(a) \subseteq r(a) \) for every \( a \in R \).

Theorem 2.2

Let \( R \) be a right MC2 ring satisfying condition (*). If every simple singular right \( R \)-module is N-flat, then \( Z(R) = (0) \).

Proof: Suppose that \( Z(R) \neq (0) \), then there exists \( 0 \neq a \in Z(R) \) such that \( a^2 = 0 \). We claim that \( Z(R) + r(a) = R \). Otherwise, there exists a maximal right ideal \( M \) such that \( Z(R) + r(a) \subseteq M \). If \( M \) is not essential, then \( M = r(e) \), \( e^2 = e \in R \). Hence \( ea = 0 \) because \( a \in r(a) \subseteq M = r(e) \). If \( eRe \neq 0 \), then \( eRe = eR \) because \( eR \) is a minimal right ideal of \( R \). Since \( a \in Z(R), eReR \subseteq Z(R) \), then \( e \in Z(R) \), which is a contradiction. Hence, \( aRe = 0 \). Since \( R \) right MC2 ring, then \( aRe = 0 \),
Let R be a ring whose every simple singular right R-module is N-flat, satisfying condition (*). Then, \( Z(R) \cap Y(R) = (0) \).

**Proof:** Suppose that \( Z(R) \cap Y(R) \neq (0) \), then there exists \( 0 \neq a \in Z(R) \cap Y(R) \) such that \( a^2 = 0 \). We claim that \( Z(R) + r(a) = R \). Otherwise, there exists a maximal essential right ideal M such that \( Z(R) + r(a) \subseteq M \). Thus, R/M is N-flat, then there exists \( b \in M \) and a positive integer n such that \( a^n \neq 0 \) and \( a^n = ba^n \). Since \( a^2 = 0 \), then \( n = 1 \), and therefore \( a = ba \), which implies that \( (1-b) \in r(a) \subseteq M \), which is a contradiction. Hence, \( Z(R) + r(a) = R \). Write \( 1 = x + y \), where \( x \in Z(R) \) and \( y \in r(a) \), then \( a = ax \). Since \( x \in Z(R) \) and \( l(x) \cap l(1-x) = 0 \), \( l(1-x) = 0 \). Thus, \( a = 0 \) because \( a \in l(1-x) \), which is a contradiction. This implies that \( Z(R) = (0) \). □

**Theorem 2.3**

Let R be a ring whose every simple singular right R-module is N-flat, satisfying condition (*). Then, \( Z(R) \cap Y(R) = (0) \).

**Lemma 2.4 [1]**

If R is a ring satisfying condition (*) whose every simple singular right R-module is N-flat, then \( J(R) \cap Y(R) = (0) \). □

According to [11], a right R-module M is said to be right weakly principally small injective (WPSI) if for any \( 0 \neq a \in J(R) \), there exists a positive integer n such that \( a^n \neq 0 \) and any right R-homomorphism from \( a^n R \rightarrow M \) can be extended to \( R \rightarrow M \). A ring R is called right WPSI if \( R_k \) is a right WPSI.

**Definition 2.5 [6]**

The ring R is said to be right(left) mininjective if every R-homomorphism from a minimal right(left) ideal of R to \( R_k \) can be extended from R to \( R_k \).

**Lemma 2.6 [11]**

If R is a left (right) WPSI ring, then:

1. \( J(R) \subseteq Z(R) \) (\( J(R) \subseteq Y(R) \)).
2. R is a left (right) mininjective ring. □

**Corollary 2.7**

Let R be a right MC2 ring, left WPSI satisfying condition (*). If every simple singular R-module is N-flat, then \( J(R) = (0) \).

**Proof:** From Lemma 2.6, we have \( J(R) \subseteq Z(R) \). So \( J(R) = (0) \) because \( Z(R) = (0) \), by Theorem 2.2. □

**Proposition 2.8**

Let R be a right WPSI ring, whose every simple singular right R-module is N-flat and satisfying condition (*). Then, \( Z(R) = (0) \).
**Proof**: By Lemma 2.4, \( J(R) \cap Y(R) = (0) \). Since \( R \) is a right WPSI ring then, \( J(R) \subseteq Y(R) \) Lemma 2.6, so \( J(R) = J(R) \cap Y(R) = (0) \) and \( R \) is a right mininjective ring Lemma 2.6. Therefore, \( R \) is a right MC2 ring [5, Theorem 1.14]. Hence, by Theorem 2.2, \( Z(R) = (0) \).

3. Certain Rings Whose every Simple Singular \( R \)-Module is \( N \)-Flat

In this section, we study the relation between rings whose every simple singular right \( R \)-modules are \( N \)-flat, reduced rings and strongly rings by adding some types of rings such as \( \text{wjc} \) and \( \text{MC2} \) rings, and other types of rings.

**Theorem 3.1**

Let \( R \) be a \( \text{wjc} \) ring satisfying condition (*) . If every simple singular right \( R \)-module is \( N \)-flat, then \( R \) is a semiprime.

**Proof**: Suppose that \( I \) is an ideal of \( R \) with \( I^2 = 0 \). If \( I \neq 0 \), then there exists \( 0 \neq a \in I \) such that \( aI = 0 \) and so \( a^2 = 0 \). First observe \( r(a) \) which is an essential right ideal of \( R \). If not, then there exists a nonzero right ideal \( K \) of \( R \) such that \( r(a) \cap K = 0 \). Since \( aK \leq aI = 0 \), \( K \subseteq r(a) \cap K = 0 \). Since \( R \) is a \( \text{wjc} \) ring, \( IK = 0 \). Hence, \( aK = 0 \) and so \( K \subseteq r(a) \cap K = 0 \), which is a contradiction. Hence, \( r(a) \) is essential and \( r(a) \neq R \). Thus, there exists a maximal essential right ideal \( M \) of \( R \) containing \( r(a) \). Hence, \( R/M \) is simple singular \( R \)-module and so is \( N \)-flat. Since \( R/M \) is \( N \)-flat, then there exists \( d \in M \) and a positive integer \( n \) such that \( a^n \neq 0 \) and \( a^n = da^n \), since \( a^2 = 0 \), then \( n = 1 \), so that \( a = da \) and we get \( (1 - d) \in l(a) \subseteq r(a) \subseteq M \), where \( 1 \in M \), which is also a contradiction. Hence, \( I = 0 \) and so \( R \) is a semiprime. □

**Definition 3.2** [13]

The ring \( R \) is called strongly right min-abel if every right minimal idempotent element \( e \in R \), is a left semicentral element.

We now consider other condition for right simple singular \( R \)-module \( N \)-flat to be semiprime.

**Theorem 3.3**

Let \( R \) be a strongly right min-abel ring satisfying condition (*). If every simple singular right \( R \)-module is \( N \)-flat, then \( R \) is a semiprime ring.

**Proof**: Let \( a \in R \) satisfies \( aRa = 0 \). Suppose that \( a \neq 0 \). Then, there exists a maximal right ideal \( M \) of \( R \) containing \( r(a) \). If \( M \) is not essential in \( R \), then \( M \) is a direct summand of \( R \), since \( M \) is maximal. So we can write \( M = r(e) \) for some \( 0 \neq e^2 = e \in R \), \( b \in R \) and hence \( eb = 0 \). Because \( eR \) is a minimal right ideal of \( R \) and \( R \) is a strongly right min-abel ring, \( be = ebe = 0 \). Thus, \( e \in r(b) \subseteq M = r(e) \), whence \( e = 0 \). This is a contradiction. Therefore, \( M \) must be an essential right ideal of \( R \). Thus, \( R/M \) is \( N \)-flat and so there exists \( d \in M \) such that \( a = da \) implies that \( (1 - d) \in l(a) \subseteq r(a) \subseteq M \) whence \( 1 \in M \). This is also a contradiction. Hence, \( R \) is a semiprime ring. □
Theorem 3.4

Let R be a strongly right min-able with \( l(a^n) \subseteq r(a) \) for every \( a \in R \) and apposite integer \( n \). If every simple singular right R-module is N-flat. Then, \( R \) is a right weakly regular ring.

Proof: We show that \( RaR + r(a) = R \) for any \( a \in R \). Suppose that there exists \( b \in R \) such that \( RbR + r(b) \neq R \). Then, there exists a maximal right ideal \( M \) of \( R \) containing \( RbR + r(b) \). If \( M \) is not essential in \( R \). Then, \( M \) is a direct summand of \( R \) because \( M \) is maximal.

By a similar method of proof used in Theorem 3.3, \( M \) is an essential right ideal of \( R \). Thus, \( R/M \) is N-flat and so there exists a positive integer \( n \) such that \( 0 \neq n \in M \) and \( n \in M \), a contradiction. Therefore, \( R \) is a right weakly regular.

According to [1], rings satisfying condition (*), and right GQ-injective whose every simple singular right \( R \)-module is N-flat are always regular. But, in general rings satisfying condition (*) and left GQ-injective whose every simple singular right \( R \)-module is N-flat need not be regular. This leads to the following theorem:

Theorem 3.5

Let \( R \) be a left GQ-injective, right MC2 ring whose every simple singular right \( R \)-module is N-flat and satisfying condition (*). Then, \( R \) is a regular ring.

Proof: First \( R \) is left GQ-injective, then \( Z(R) = J(R) \) and \( R/J(R) \) is a regular ring. From Theorem 2.2, \( Z(R) = (0) \). Hence, \( J(R) = Z(R) = (0) \), which implies \( R \) is a regular ring.

Definition 3.6 [2]

\( R \) is called a right CAM-ring, if for any maximal essential right ideal \( M \) of \( R \) (if it exists) and for any right subideal \( I \) of \( M \) which is either a complement right subideal of \( M \) or a right annihilator ideal in \( R \), \( I \) is an ideal of \( M \).

Right CAM-ring generalizes semisimple artinian. [2] shows that semiprime right CAM-ring is either semisimple artinian or reduced. If \( R \) is also simple singular right \( R \)-module is N-flat, then \( R \) is either semisimple artinian or strongly regular ring. We yield the following theorem:

Theorem 3.7

Let \( R \) be a wjc right CAM-ring, satisfying condition (*) whose every simple singular right \( R \)-module is N-flat. Then, \( R \) is either a semisimple artinian or strongly regular ring.

Proof: From Theorem 3.1 \( R \) is semiprime. If \( R \) is not a semisimple artinian, then \( R \) is reduced. Let \( 0 \neq a \in R \). If \( aR \oplus r(a) \neq R \), then \( aR \oplus r(a) \subseteq M \) for some maximal right ideal \( M \) of \( R \). If \( M \) is not an essential right ideal of \( R \), then \( M = eR \), where \( e^2 = e \in R \). Because \( R \) is reduced, \( ae = ea = 0 \) and \( e \in r(a) \subseteq M = r(e) \), a contradiction. Hence, \( M \) is an essential right ideal of \( R \) and so \( R/M \) is a singular simple right \( R \)-module. Since \( R/M \) is N-flat, then there exists \( b \in M \) and a positive integer \( n \) such that \( a^n \neq 0 \) and \( a^n = ba^n \). Now, we obtain \( (1-b) \in r(a) \), so \( 1 \in M \), a contradiction. Hence, \( aR \oplus r(a) = R \) and then \( R \) is a strongly regular ring.

An idempotent \( e \in R \) is called right semicentral if \( ea = eae \) for all \( a \in R \) [13].
Lemma 3.8 [12]
Let R be a MERT ring and every right minimal idempotent in \( R/J(R) \) is right semicentral. Then, \( R/J(R) \) is reduced.  

Proposition 3.9
Let R be a MERT ring and every right minimal idempotent of \( R/J(R) \) is right semicentral. If every simple singular right \( R \) -module is \( N \) -flat. Then, \( R/J(R) \) is a strongly regular ring.

Proof: By Lemma 3.8, \( R/J(R) \) is reduced. Let \( 0 \neq a \in R \) \( R/J(R) \). We first that \( \bar{R}aR + r(\bar{a}) = \bar{R} \). Suppose that it is not; then there exists a maximal right ideal M of \( R \) such that \( \bar{R}aR + r(\bar{a}) \subseteq M/J(R) \). Since \( R' \) is reduced, \( \bar{R}aR + r(\bar{a}) \) is an essential right ideal of \( R' \). Therefore, \( R/M \) is simple singular right \( R \)-module and so \( R/M \) is \( N \)-flat.

Now, it is easy to show that there exists a positive integer \( n \) and \( c \in M \) such that \( a^n \neq 0 \) and \( a^n = ca^n \). Since \( M \) is an ideal of \( R \), we have \( 1 \in M \), which is a contradiction. Hence, \( \bar{R}aR + r(\bar{a}) = \bar{R} \). Let \( \bar{a} \in \bar{R}aR \), where \( \bar{a} \in \bar{R}aR \). Then, \( (\vec{a} - \vec{d}) \in \bar{R} \). If \( \bar{a}R + r(\bar{a}) \neq \bar{R} \), then there exists a maximal right ideal \( L \) of \( R \) such that \( \bar{a}R + r(\bar{a}) \subseteq L/J(R) \). Since \( R/J(R) \) is reduced, we have \( L/J(R) \) is an ideal of \( R/J(R) \). Hence, \( L \) is also an ideal of \( R \), and so \( \bar{a}R + r(\bar{a}) \subseteq L/J(R) \), and hence \( \bar{a}R + r(\bar{a}) \subseteq L/J(R) \), which is also a contradiction. Thus, \( \bar{a}R + r(\bar{a}) = \bar{R} \), and so \( R/J(R) \) is strongly regular ring. 

Lemma 3.10 [4]
For any \( a \in Cent(R) \) (The Center of \( R \)), if \( a = ara \) for some \( r \in R \); Then there exists \( b \in Cent(R) \) such that \( a = aba \).

Lemma 3.11 [4]
If \( R \) is a semiprime ring; then \( r(a^n) = r(a) \) for any \( a \in Cent(R) \) and a positive integer \( n \).

Proposition 3.12
Let \( R \) be a wjc ring satisfying condition (*) . If every simple singular right \( R \)-module is \( N \)-flat. Then, the Center of \( R \) is a regular ring.

Proof: Let a be a nonzero element in Cent(R). First, we will show that \( aR + r(a) = R \) for any \( a \in Cent(R) \). If not, there exists a maximal right ideal \( M \) of \( R \) such that \( aR + r(a) \subseteq M \). Since \( a \in Cent(R) \), \( aR + r(a) \) is an essential right ideal and so \( M \) must be an essential right ideal of \( R \). Therefore, \( R/M \) is \( N \)-flat, so there exists \( c \in M \) and a positive integer \( n \) such that \( a^n \neq 0 \) and \( a^n = ca^n = a^n c \) implies that \( 1 - c \in r(a^n) \subseteq r(a) \subseteq M \) (Theorem 3.1 and Lemma 3.11) and so \( 1 \in M \), which is a contradiction. Therefore, \( aR + r(a) = R \) for any \( a \in Cent(R) \) and so we have \( a = ara \) for some \( r \in R \). Applying Lemma 3.10, Cent(R) is a regular ring.

Theorem 3.13
\( R \) is strongly regular ring if and only if \( R \) is a wjc, MERT and 2 - primal ring whose every simple singular right \( R \)-module is \( N \)-flat.
**Proof**: First, we show that $R$ is reduced. In fact, if $a^2 = 0$ for some $0 \neq a \in R$. Then, we have $RaR + r(a) = R$. If not, then there exists a maximal right ideal $M$ of $R$ containing $RaR + r(a)$. Observe that $M$ must be an essential right ideal of $R$. If not, then $M$ is a direct summand of $R$. So we can write $M = r(e)$ for some idempotent $e$ of $R$. Thus $eRa = 0$. Since $R$ is a gjc ring, $aRe = 0$ and $ae = 0$. Hence, $e \in r(a) \subseteq r(e)$; whence $e = 0$, it is a contradiction. Therefore, $M$ must be an essential right ideal of $R$. Then, $R/M$ is $N$-flat, there exists a positive integer $n$ and $b \in M$ such that $a^n \neq 0$ and $a^n = ba^n$. Since $a^2 = 0$, then $n = 1$ and therefore $a = ba$ which implies $(1 - b) \in M$ and so $1 \in M$ by $M$ is an ideal of $R$. This is a contradiction. Hence $RaR + r(a) = R$ and so $a = ad$ for some $d \in RaR$. Since $R$ is 2-primal ring, $d \in J(R)$. Hence, $(1 - d)$ is right invertible $v$ in $R$ such that $v(1 - d) = 1$, $v(a - da) = a$ which yield $a = 0$.

Next, we show that $aR + r(a) = R$ for each $a \in R$. If not, then there exists $b \in R$ and a maximal right ideal $L$ of $R$ containing $bR + r(b)$. Observe that $L$ must be essential, so there exists a positive integer $n$ and $d \in M$ such that $a^n \neq 0$ and $a^n = da^n$. Now, $(1 - d) \in l(a^n) = r(a^n) = r(a) \subseteq L$ so $1 \in L$, which is a contradiction; therefore $aR + r(a) = R$. Hence, $R$ is strongly regular ring.

**Conversely**: it is obvious. □
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